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E2 Parametric identification with Least-Squares method

This exercise is dedicated to parametric identification using the least-squares (LS) method. During the exercise, an assumption will be made that a structure of the considered system is known a priori, i.e. we will apply the GrayBox approach. When we use the batched-type LS estimator all the measurements from the input and output of a plant are simultaneously used for estimation of model parameters.

1 Static plant identification using LS method

By the *static plant* we will understand:

- a system with no dynamics, i.e. the steady-state response of the plant appears on the plant output instantly after an input signal u is applied (with no transient states), or
- a static relation between the input signal and the plant output at a steady state for a given dynamic plant (a transient response of the plant is not relevant to us).

An exemplary static relation (given as a set of data) between the input and the output of a plant is presented in Fig. 1. The static plant can be represented by an algebraic mapping

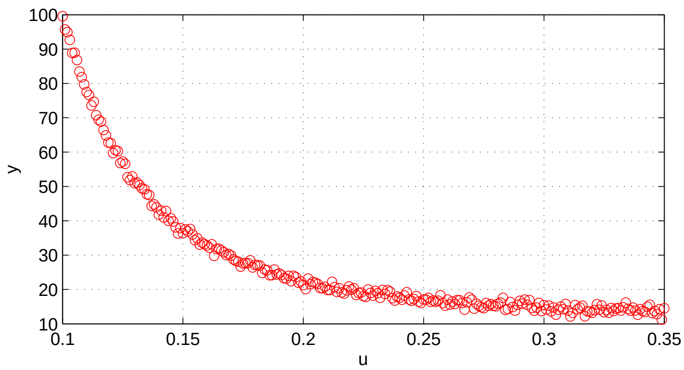

Figure 1: Static relation between the applied input u and the measured output y of a plant at a steady state.

(either linear or nonlinear) in the following general form:

$$u \mapsto y \quad \Rightarrow \quad y = f_0(u, \boldsymbol{p}_0) + v,$$

where $\boldsymbol{p}_0 = [p_{10}\ p_{20} \dots p_{d0}]^\top$ is a vector of *true* parameters of the plant, $f_0(u, \boldsymbol{p}_0)$ is a *true* description of the plant for the noise-free conditions, while v represents a noise term which is always present in practical scenarios. The parametrization of function $f_0(\cdot)$ can be performed using various approaches. In this exercise, we will be focused on the linear parametrizations which lead to a model in the linear regression form , i.e. the model can be expressed as a linear combination of some parameters $p_i,\ i = 1, 2, \dots, d$ and a chosen base functions:

$$f(u, \boldsymbol{p}) \triangleq \sum_{i=1}^{d} p_i \cdot F_i(u) \qquad \Rightarrow \qquad \hat{y} = \sum_{i=1}^{d} p_i \cdot F_i(u). \tag{1}$$

Model (1) can be written in the linear regression form as follows:

$$\hat{y} = \boldsymbol{\varphi}^\top(u)\boldsymbol{p} \qquad \Rightarrow \qquad y := \boldsymbol{\varphi}^\top(u)\boldsymbol{p} \tag{2}$$

where $\boldsymbol{\varphi}(u) = [F_1(u)\ F_2(u)\ \ldots\ F_d(u)]^\top$ is the regression vector depending on a deterministic input u (through base functions $F_i(u)$). Application of the least-squares method to equation errors $\varepsilon_n(\boldsymbol{p}) \triangleq y_n - \boldsymbol{\varphi}_n^\top(u)\boldsymbol{p}$ written for the right-hand equation in (2) for $n \in [1, N]$ leads to the following LS estimator being the unique solution to the parametric identification problem:

$$\hat{\boldsymbol{p}}_N^{\mathrm{LS}} = (\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{y}, \tag{3}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}_1^\top(u) \\ \vdots \\ \boldsymbol{\varphi}_N^\top(u) \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \tag{4}$$

and $\boldsymbol{\Phi}$ is (from assumption) a deterministic regression matrix depending on the deterministic input signal u. A confidence level of the estimate[1] computed by (3) is strictly related to the covariance matrix $\mathrm{cov}[\hat{\boldsymbol{p}}_N^{LS}]$, which for the case where v is a white noise can be estimated based on the measurement data as follows:

$$\mathrm{cov}[\hat{\boldsymbol{p}}_N^{LS}] \approx \hat{\boldsymbol{P}}_N = \hat{\sigma}^2(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1} \qquad \hat{\sigma}^2 = \frac{1}{N-d}\sum_{i=1}^{N}\varepsilon_i^2(\hat{\boldsymbol{p}}_N^{\mathrm{LS}}), \tag{5}$$

where $\hat{\sigma}^2$ is the estimate of variance of noise v, $\varepsilon_i(\hat{\boldsymbol{p}}_N^{\mathrm{LS}}) = y_i - \boldsymbol{\varphi}_i^\top(u)\hat{\boldsymbol{p}}_N^{\mathrm{LS}}$ is the equation error computed for estimated parameters, while d is a number of parameters used in the model. In the case where disturbance v is a colored noise, the computation of the covariance matrix $\mathrm{cov}[\hat{\boldsymbol{p}}_N^{LS}]$ requires the knowledge of a full covariance matrix of noise v.

1.1 Identification of a static plant.

- File `IdentStat.mat` contains two sets of measurement data (pairs (u, y)), gathered from a static plant and stored in matrices `DataStatW, DataStatC`, where the first one contains measurements corrupted by a white noise, while the second one corrupted by a colored noise. Load the data from the file to the Matlab workspace using command `load IdentStat.mat`.
- Assuming the following structure of the static model:

$$\hat{y} = p_1 + \sum_{i=2}^{4}\frac{p_i}{u^{i-1}} \tag{6}$$

  compute the estimates of model parameters using equation (3). The calculations should be done independently for the data with a while noise and for the data with a colored noise.
- Plot the measurement data and the identified function (6) on a common figure. Upon the plots, evaluate quality of the performed identification.
- Verify the influence of the amount of measurement data on the identification quality (to this aim use different subsets of measurement data).
- Estimate covariance matrix (5) for the case of data corrupted by a white noise and assess confidence levels for particular estimates of parameters.

[1] Remember that $\hat{\boldsymbol{p}}_N$ is a random variable!

2 Dynamic plant identification using LS method

Let us consider a following model of a dynamical plant in the discrete-time domain:

$$A(\overline{q},\boldsymbol{p})y(n) = B(\overline{q},\boldsymbol{p})u(n) + e(n) \qquad \Rightarrow \qquad y(n) = G(\overline{q},\boldsymbol{p})u(n) + v(n), \tag{7}$$

where $v(n) = H(\overline{q},\boldsymbol{p})e(n)$, $e(n)$ is a white noise (by assumption), $G(\overline{q},\boldsymbol{p}) = \frac{B(\overline{q},\boldsymbol{p})}{A(\overline{q},\boldsymbol{p})}$ and $H(\overline{q},\boldsymbol{p}) = \frac{1}{A(\overline{q},\boldsymbol{p})}$ are the transfer function operators representing the control-route and noise-route dynamics, respectively, while $A(\overline{q},\boldsymbol{p})$ and $B(\overline{q},\boldsymbol{p})$ are the polynomials of degree n_a and n_b, respectively. The aim is to identify parameters $\boldsymbol{p}$ of the transfer function operator $G(\overline{q},\boldsymbol{p})$ by using the LS method. The structure of model (7) allows writing the output of the system (and consequently the equation error) as a linear function of parameters:

$$y(n) = \boldsymbol{\varphi}^\top(n)\boldsymbol{p} + e(n), \qquad \varepsilon(n,\boldsymbol{p}) \triangleq y(n) - \boldsymbol{\varphi}^\top(n)\boldsymbol{p}, \tag{8}$$

where the regressor

$$\boldsymbol{\varphi}^\top(n) = [-y(n-1)\ \dots\ -y(n-n_a)\ u(n-1)\ \dots\ u(n-n_b)]^\top \tag{9}$$

is no longer a deterministic function, but a stochastic one (as the result of using the auto-regression model). Application of the LS method to equation errors (8) for $n \in [1, N]$ leads to the following LS estimator, being a unique solution to the parametric identification problem:

$$\hat{\boldsymbol{p}}_N^{\mathrm{LS}} = (\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{y}, \tag{10}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}^\top(1) \\ \vdots \\ \boldsymbol{\varphi}^\top(N) \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}, \tag{11}$$

whereas this time $\boldsymbol{\Phi}$ is a stochastic regression matrix depending on the previous samples of output y and deterministic input u. In this case, the covariance matrix can be estimated using

$$\mathrm{cov}[\hat{\boldsymbol{p}}_N^{LS}] \approx \frac{1}{N}\hat{\boldsymbol{P}}_\infty = \frac{1}{N}\hat{\sigma}^2\left(\frac{1}{N}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1} = \hat{\sigma}^2\left(\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1}, \qquad \hat{\sigma}^2 = \frac{1}{N-d}\sum_{n=1}^{N}\varepsilon^2(n,\hat{\boldsymbol{p}}_N^{\mathrm{LS}}), \tag{12}$$

where $\varepsilon(n,\hat{\boldsymbol{p}}_N^{\mathrm{LS}}) = y(n) - \boldsymbol{\varphi}^\top(n)\hat{\boldsymbol{p}}_N^{\mathrm{LS}}$ is the equation error computed for estimated parameters, while d is a number of parameters used in the model. If $e(n)$ is in fact a white noise (i.e. if the formal assumption of a nature of $e(n)$ is satisfied in practice) and matrix $\boldsymbol{\Phi}^\top\boldsymbol{\Phi}$ is non-singular, then estimator (10) is consistent, that is $\lim_{N\to\infty}(\hat{\boldsymbol{p}}_N^{\mathrm{LS}} - \boldsymbol{p}_0) = \boldsymbol{0}$ (convergence with probability equal to 1).

Auxiliary information regarding the data in file `IdentDyn.mat`:

- matrices: `DataDynW=[u yw], DataDynC=[u yc]`
- simulation time: `t=0:Tp:100`, sampling time `Tp=0.05` s
- true parameters of the plant: $k_0 = 2.0$, $T_0 = 0.5$ (unknown in practice)
- applied input signal: $u(t) = 0.2\sin(5t) + 0.1\sin(2t) + 0.5\cos(2t)$
- method of white noise generation for ARX structure: $v = H(\overline{q})e$, where $H(\overline{q}) = 1/A(\overline{q})$, hence `H=tf([1 0],[1 -exp(-Tp/T)],1)`, $e = 0.1 \cdot$`randn(N,1)`
- method of colored noise generation for ARX structure: $v = H_1(\overline{q})e$, where `H1=tf([1 1.5 1.1],[1 0 0],1)`, $e = 0.1 \cdot$`randn(N,1)`

2.1 Parametric identification of the ARX model structure using LS method.

- Consider the following first-order dynamic plant:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k_0}{T_0 s + 1} \qquad \Rightarrow \qquad G(z) = \frac{Y(z)}{U(z)} = \frac{k(1 - e^{-T_p/T})}{z - e^{-T_p/T}}, \tag{13}$$

  with some unknown parameters k_0 and T_0, where $G(z)$ is the discrete approximation model of $G(s)$ obtained using the zero-order-hold (`zoh`) method. By using the above transfer function, we can write the discrete-time model of the plant with operator $\overline{q} = q^{-1}$ as $y(n) = G(\overline{q}, \boldsymbol{p})u(n) + v(n)$, where $v(n)$ is a noise term.

- Rewrite the plant model (13) in a linear regression form and find the regression vector and the vector of parameters for the model.

- File `IdentDyn.mat` contains two sets of measurement data (pairs $(u(n), y(n))$) stored in matrices `DataDynW, DataDynC`, where the first one contains measurements corrupted by a white noise, while the second one contains data corrupted by a colored noise. Load the data from the file to the Matlab workspace using command `load IdentDyn.mat`. Divide the data into two subsets (e.g. in the proportion of 50/50 percent): Z_{est} which will be used for model estimation, and Z_{val} which will be used for model validation.

- By assuming the ARX model structure perform the parametric identification procedure of model (13) using equation (10) with data from subset Z_{est}. Calculations should be done independently for the case of data corrupted by a white noise and for data corrupted by a colored noise.

- Based on vector $\hat{\boldsymbol{p}}_N^{\text{LS}}$, calculate estimates $\hat{k}$ and $\hat{T}$.

- In a common figure plot the measured plant response $y(n)$ from subset Z_{val} and the response $y_m(n)$ of the identified simulated model excited with input signal $u(n)$ taken from subset Z_{val}. Evaluate quality of parametric identification by analyzing the plots. Next, evaluate quality of identification by computing the following index

$$V(\hat{\boldsymbol{p}}_N^{\text{LS}}) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} \varepsilon^2(n, \hat{\boldsymbol{p}}_N^{\text{LS}}), \tag{14}$$

  where $\varepsilon(n, \hat{\boldsymbol{p}}_N^{\text{LS}}) := y(n) - \hat{y}(n|n-1)$, $\hat{y}(n|n-1)$ is the output of the one-step-ahead predictor, while N denotes now the amount of data included in subset Z_{val}.

- Compute covariance matrix (12) for the case of data corrupted by a white noise and assess confidence levels for particular estimates of parameters.

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