Laboratory of Adaptive Control

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E5 Model-Reference Adaptive Control (MRAC)

The exercise is devoted to the adaptive control design problem in the MRAC (Model-Reference Adaptive Control) scheme for the exemplary plant and to verification of the designed control system in the Matlab-Simulink environment. For the controller parameters we will apply adjustment rules derived upon the Lyapunov stability analysis which guarantees boundedness and asymptotic convergence of the model-following error in the MRAC control system.

1 Description of the plant

Let us consider the *aero-plant* in a form of the helicopter pitch dynamics in hover as illustrated in Fig. 1. By tilting of the main rotor it is possible to force the elevator-like input δ

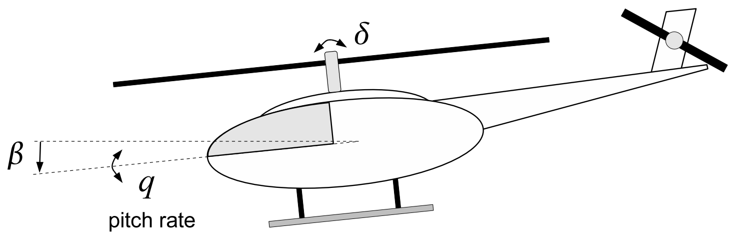

Figure 1: Hovering helicopter in the pitch motion forced by the elevator-like input δ (based on E. Lavretsky, K. A. Wise: Robust and Adaptive Control with Aerospace Applications, Springer, London, 2013)

to pitch dynamics of the helicopter influencing the pitch rate q. Locally, and neglecting some non-dominating effects, the dynamics of pitch motion can be approximated by the following equation

$$\dot{q}(t) = \theta_{a0}\, q(t) - \theta_{b0}\, \delta(t) + \theta_{c0}\ \mathrm{tgh}\left(\frac{360}{\pi} q(t)\right), \tag{1}$$

where q denotes the pitch rate in [rad/s], δ is the control input in [rad], while θ_{a0}, θ_{b0}, and θ_{c0} represent the unknown (true) parameters of the pitch dynamics. Note that according to physical interpretation of the control input, a practical range of absolute values for δ shall be limited to several angular degrees.

2 Control performance requirements

We are going to design the MRAC system for pitch dynamics represented by (1) with state $x \triangleq q$ and control input $u \triangleq \delta$ which guarantees satisfaction of the following prescribed performance requirements:

R1. pitch rate $q(t)$ follows a time-varying reference $x_m(t) = q_m(t)$ in the form of a bounded time-varying signal with bounded time-derivative, R2. the following-error

$$e(t) \triangleq x(t) - x_m(t) \tag{2}$$

asymptotically converges to zero, that is: $\lim_{t\to\infty} e(t) = 0$.

3 Control system design

3.1 Step 1: design of the reference model

In the MRAC scheme, the control performance requirements can be satisfied by appropriately designing the reference model and the controller. Let us focus now on the reference model which will be designed for the linear part of aero-plant (1) by defining the auxiliary first-order dynamics

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t), \tag{3}$$

where $a_m < 0$ and $b_m > 0$ are the design (constant) parameters, while $r(t)$ denotes the reference command resulting from a particular desired pitch motion of the helicopter. It is well known that for $a_m < 0$ solution to (3) in response to any bounded command $r(t)$ is bounded and has bounded time-derivative (requirement R1). Reference model (3) applied in the MRAC scheme determines transient performance for the process state $x(t) = q(t)$. In particular, for the step command of amplitude R and zero initial conditions we have

$$x_m(t) = R\frac{b_m}{-a_m}(1 - \exp(a_m t)) \tag{4}$$

which for $a_m < 0$ asymptotically converges to value $\frac{b_m}{-a_m}R$.

3.2 Step 2: reformulation of process dynamics

Recalling the general form of the plant assumed in the Lyapunov-based MRAC control

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_0\boldsymbol{x}(t) + \bar{\boldsymbol{B}}_0\boldsymbol{\Lambda}_0[\boldsymbol{u}(t) + \boldsymbol{\Theta}_0^\top\boldsymbol{\Phi}(\boldsymbol{x}(t))] \tag{5}$$

one shall first reformulate the original pitch dynamics (1) to conform to the general description (5). To this aim, let us rewrite (1) as follows:

$$\dot{x}(t) = \underbrace{\theta_{a0}}_{a_0} x(t) + \underbrace{(-1)}_{\bar{b}_0} \cdot \underbrace{\theta_{b0}}_{\lambda_0} \left(u(t) + \underbrace{\frac{-\theta_{c0}}{\theta_{b0}}}_{\theta_0} \underbrace{\mathrm{tgh}\left(\frac{360}{\pi}x(t)\right)}_{\phi(x(t))} \right), \tag{6}$$

where $x(t) \triangleq q(t)$ is a one-dimensional state of the process (due to the first-order dynamics (1)), and $u(t) \triangleq \delta(t)$ is a scalar control input. Note: for the first-order plant we consider a special (simplified) case where all the terms $\boldsymbol{A}_0 = a_0$, $\bar{\boldsymbol{B}}_0 = \bar{b}_0$, $\boldsymbol{\Lambda}_0 = \lambda_0$, $\boldsymbol{\Theta}_0 = \theta_0$, and $\boldsymbol{\Phi}(\boldsymbol{x}) = \phi(x)$ are the scalars. The characteristic components distinguished in equation (6) will be used in Section 3.3.

Worth stressing the negative sign of control effectiveness characteristic to plant (6) and represented by component $\bar{b}_0$ which, we assume, is perfectly known upon the a priori knowledge.

3.3 Step 3: design of the controller block and adjustment rules

First, we shall check the so-called *matching conditions* to find out if the nominal MRAC controller of the general form

$$\boldsymbol{u}(t) \triangleq \boldsymbol{K}_0 x(t) + \boldsymbol{L}_0 r(t) - \boldsymbol{\Theta}_0^\top\boldsymbol{\Phi}(\boldsymbol{x}(t)) \tag{7}$$

does exist for the plant dynamics (6). In the considered first-order dynamics (6) the general form (7) reduces to

$$u(t) \triangleq k_0 x(t) + l_0 r(t) - \theta_0 \phi(x(t)), \qquad \text{where} \qquad k_0, l_0, \theta_0 \in \mathbb{R}. \tag{8}$$

The matching conditions for the controller (8) and plant described by (5) take the following form:

$$\boldsymbol{A}_0 + \bar{\boldsymbol{B}}_0 \boldsymbol{\Lambda}_0 \boldsymbol{K}_0^\top = \boldsymbol{A}_m \quad \overset{(6),(8)}{\Longrightarrow} \quad \theta_{a0} + (-1) \cdot \theta_{b0} k_0 = a_m, \tag{9}$$

$$\bar{\boldsymbol{B}}_0 \boldsymbol{\Lambda}_0 \boldsymbol{L}_0^\top = \boldsymbol{B}_m \quad \overset{(6),(8)}{\Longrightarrow} \quad (-1) \cdot \theta_{b0} l_0 = b_m. \tag{10}$$

According to the above equations one can conclude that the nominal controller (8) exists for the nominal gains

$$k_0 = \frac{a_m}{\theta_{a0} - \theta_{b0}}, \qquad l_0 = \frac{b_m}{-\theta_{b0}}. \tag{11}$$

Obviously, the nominal controller (8) with gains (11) cannot be applied in practice because true plant parameters θ_{a0}, θ_{b0}, and θ_{c0} are unknown. Therefore, we introduce the adjustable (adaptive) version of the controller

$$u(t) \triangleq \hat{k}(t) x(t) + \hat{l}(t) r(t) - \hat{\theta}(t) \phi(x(t)), \tag{12}$$

where the estimates of gains $\hat{k}(t)$, $\hat{l}(t)$, and $\hat{\theta}(t)$ need to be adjusted on-line in a way which guarantee satisfaction of prescribed requirement R2. It can be shown, by applying the Lyapunov stability criterion for the closed-loop system comprising plant (6) and controller (12), that the *adjustment rules* shall be formulated as follows

$$\hat{k}(t) = \hat{k}(0) - \frac{p\bar{b}_0}{\gamma_x} \int_0^t x(\tau) e(\tau) d\tau, \tag{13}$$

$$\hat{l}(t) = \hat{l}(0) - \frac{p\bar{b}_0}{\gamma_r} \int_0^t r(\tau) e(\tau) d\tau, \tag{14}$$

$$\hat{\theta}(t) = \hat{\theta}(0) + \frac{p\bar{b}_0}{\gamma_\phi} \int_0^t \phi(x(\tau)) e(\tau) d\tau, \tag{15}$$

where model-following error $e(t)$ has been defined by (2), γ_x, γ_r, and γ_ϕ are inverses of positive adaptation gains selected by a designer (they influence the adaptation rate), whereas p is a positive-definite (constant) solution to the Lyapunov equation

$$\boldsymbol{A}_m^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_m = -\boldsymbol{I} \quad \overset{(3)}{\Longrightarrow} \quad a_m p + p a_m = -1 \quad \Rightarrow \quad p = \frac{-1}{2a_m}. \tag{16}$$

The resultant MRAC control system corresponding to control law (12) is presented in Fig. 2.

3.1 Adaptive control of the aero-plant in the MRAC system.

- Open the file `HelicPlantMRAC.mdl` which contains the aero-plant (1), reference model (3), and the reference command generator (RCG). The RCG block produces two types of the reference command $r(t)$ and its time derivative $\dot{r}(t)$:

$$\text{TYPE 1:} \quad r(t) \triangleq R\sin(\omega_r t), \qquad \dot{y}_r(t) = R\omega_r \cos(\omega_r t), \tag{17}$$

$$\text{TYPE 2:} \quad r(t) \triangleq R\,\mathrm{rect}(\omega_r t), \qquad \dot{y}_r(t) = 0, \tag{18}$$

  where $\mathrm{rect}(\omega_r t)$ represents a symmetric rectangular signal with unit amplitude and frequency ω_r rad/s.

- Initialize the following global variables: `Tc`=0.1 s, `sigma2e`=0.0, which represent, respectively, the sampling interval and the variance for a stochastic noise generator disturbing the plant. Further, initialize parameters of the reference model taking $a_m = -4.0$ and $b_m = 4.0$.

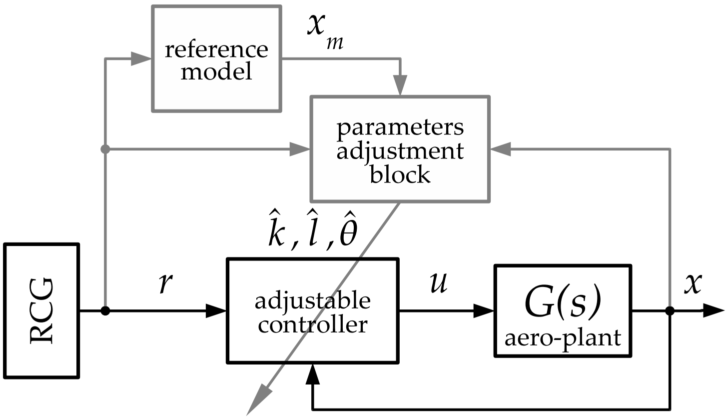

Figure 2: Block scheme of the MRAC system for the aero-plant; the blocks and arrows highlighted in gray indicate components of the adaptive loop, while the black ones correspond to the conventional part of the control system (RCG = Reference Command Generator)

- Implement the parameters adjustment block (see Fig. 2) according to the adjustment rules (13)-(15). Initially prescribe $\gamma_x = 0.0005$, $\gamma_r = 0.0005$, and $\gamma_\phi = 0.1$.
- Implement the adjustable controller (12) with $\phi(x(t))$ taken from model (6).
- Run the MRAC control system and analyze the resultant control quality for both types of a reference command generated by the RCG block – see (17)-(18) – using parameters: $R = 0.15\,\mathrm{rad/s}$, $\omega_r = 0.25\,\mathrm{rad/s}$.
  Important: for the analysis purposes check the time plots of the model-following error $e(t)$ as well as the control input $u(t)$, and compare state $x(t)$ with reference state $x_m(t)$ and command signal $r(t)$ on the same plot; check also time plots of estimated controller parameters $\hat{k}(t)$, $\hat{l}(t)$, and $\hat{\theta}(t)$.
  Repeat simulations for
  $$\texttt{sigma2e} \in \{0.0;\ 0.001;\ 0.01\}. \tag{19}$$
  Does the system satisfy performance requirements R1 and R2 in all the cases?
- Check influence of inverted gains γ_x, γ_r, and γ_ϕ on the control performance.
- Check influence of the reference model parameters, a_m and b_m, on the control performance.
- Introduce an auxiliary feedback signal to the reference model defining
  $$\dot{x}_m(t) = a_m x_m(t) + b_m r(t) + \underbrace{k_e(x(t) - x_m(t))}_{\text{auxiliary signal}}, \qquad k_e > 0 \tag{20}$$
  in order to obtain the so-called *observer-like reference model*; k_e in (20) is a design parameter. Selecting $k_e = 10$, run again the MRAC control system and analyze the resultant control quality for both types of a reference command generated by the RCG block. Compare the control performance with the case where the reference model has classical form (3) – take into account especially transient states of the adaptation process. Repeat simulations and analysis of the results for
  $$\texttt{sigma2e} \in \{0.0;\ 0.001;\ 0.01\}. \tag{21}$$

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