Motion control with minimization of a boundary off-track for non-Standard N-Trailers along forward-followed paths

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***Abstract*— When a multi-body tractor-trailers (N-trailer) vehicle has to follow a varying-curvature reference path, the unavoidable effect of a non-zero off-track occurs for most of the vehicle segments. The boundary off-track determines a minimal width of an envelope around the path which must be kept obstacle-free to ensure safe motion of a vehicle. The cascade-like control strategy presented by the author in his previous works is generalized here to enable minimization of the envelope width by introducing and utilizing the so-called *virtual guidance point* of a vehicle being a linear weighted combination of postures of all the vehicle segments. Application of the proposed control strategy with appropriately chosen weights allows reducing the boundary off-track for the N-trailers equipped with off-axle hitches. Exemplary simulation results show efficacy and utility of the method for a vehicle with three trailers and mixed signs of hitching offsets.**

I. Introduction

The autonomous path-following task belongs to classical motion problems formulated for mobile robots [10]. A definition of the task is usually stated with respect to the so-called *guidance point* located on a vehicle in a specific place [17]. The concept has been adopted also to the multi-body N-trailer robots (shortly: N-trailers, see Fig. 1) comprising a tractor and an arbitrary number of N trailers interconnected by passive rotary joints [18], [6], [7], [13]. In this case, the guidance point has been mostly selected as a midpoint of a wheels axle located on a last-trailer or on a tractor segment. Such an approach is especially justified in tasks where the guidance point reflects location of a working tool or a working place expected to follow a reference path as accurate as possible. An unavoidable side effect of such a task formulation is the presence of so-called *off-tracks* appearing in motion of other vehicle segments (except the guidance one). The off-tracks are especially large and time-varying in the case when a long vehicle follows a reference path of a high and variable curvature [8], [1].

Yet, many practical applications enforce slightly different formulation of the path-following objective when the long multi-body vehicles are considered. Instead of expecting accurate reproduction of the path by a single guidance point, one often requires only approximate reproduction of the path, however by all the vehicle segments. Their positions should be kept within some envelope around the path of assumed (small enough) or available width to guarantee safe (collision-free) motion conditions for all the vehicle bodies. It seems to be a natural motion objective when one considers N-trailer vehicles following public roads. In this context, location of a single guidance point on a last trailer or on a tractor segment may be undesirable, because it usually leads to the envelope of unacceptably large width. Some researchers tried to minimize the off-tracks by modifying the N-trailer mechanical structure introducing e.g. the sliding kingpin mechanism [12] or steering ability for wheels of the trailers [15], [16]. Reduction of the off-tracks for the conventional N-trailers (with non-steerable wheels and fixed kingpin hitches) by designing an appropriate feedback control strategy was addressed for the first time by Altafini, who considered in [3] and [4] autonomous path-following 'with reduced off-tracking' for the N-trailer robots moving forward. He proposed two locally valid solutions for the Generalized N-Trailer (GNT, see [2]) based on the input-output linearization concept: simple PD-like controller for the case of constant-curvature paths [4], and much more complex control law for the case of varying-curvature paths [3]. The path-following task has been defined there as an output stabilization problem with the output defined as a sum of signed orthogonal distances of all the vehicle segments to the reference path (see also [9]). To the author's knowledge, the works of Altafini presents the only feedback control solutions to the path-following problem with a reduced off-track available in the literature so far.

An objective of this article is the generalization and adaptation of the cascade-like control strategy recently presented in [13] to the task of forward path-following with reduced boundary off-track for the non-Standard N-Trailers (nSNTs, see [11]) equipped solely with off-axle hitching, which were excluded from solutions presented in [3] and [4]. In contrast to [3] and [4], the new approach does not require to compute/measure the shortest distances of vehicle segments to a reference path. Thus, the novel method removes limitations of an initial vehicle configuration imposed by solutions proposed in [3] and [4]. Thanks to the cascade-like approach, the new control law is highly scalable with respect to a number of trailers also in the case of varying-curvature reference paths (contrary to [3]). The key point of the concept comes from introduction and utilization in the feedback controller of the so-called *virtual guidance point* being a weighted linear combination of postures of all the vehicle segments. By selecting the weights, one can flexibly affect the boundary off-track of a vehicle along a reference path. As a consequence, minimization of the envelope width around the path needed to guarantee collision-free motion

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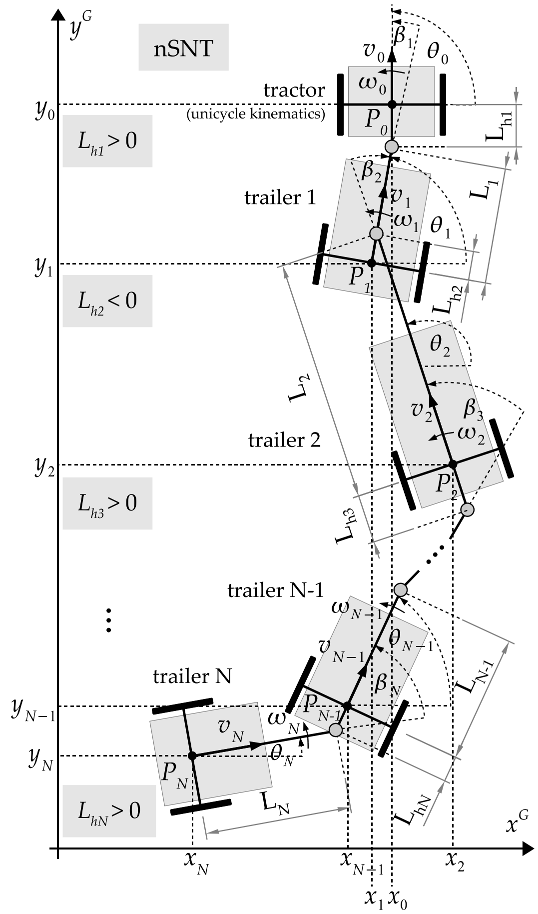

Fig. 1. Kinematic structure of the nSNT vehicle ($L_{hi} \neq 0$ for all $i = 1, \ldots, N$) with mixed signs of hitching offsets (here $L_{h2} < 0$)

for a vehicle becomes possible. The paper presents formal definition of the control problem, derivation of the generalized cascade-like control law, and illustrates influence of the weights on the obtainable boundary off-track in the closed-loop system.

II. Kinematics and problem formulation

A. Kinematics of N-trailers with non-zero hitching offsets

Kinematic structure in Fig. 1 presents the nSNT vehicle in the global frame $\{x^G, y^G\}$. The vehicle comprises of $N+1$ segments: a tractor (indexed by zero) and a number of N trailers interconnected by passive rotary joints. Position of every ith vehicle segment, $i = 0, \ldots, N$, is determined by the midpoint $P_i(x_i, y_i)$ of the wheels axle, while its orientation by angle θ_i. Assuming the rolling-without-skidding motion condition for all the vehicle wheels (represented by the nonholonomic constraint $\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$ for $i = 0, \ldots, N$), one can consider every ith vehicle body as unicycle

$$\dot{\boldsymbol{q}}_i = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta_i \\ 0 & \sin\theta_i \end{bmatrix} \begin{bmatrix} \omega_i \\ v_i \end{bmatrix} = \boldsymbol{G}(\boldsymbol{q}_i)\boldsymbol{u}_i, \quad \boldsymbol{q}_i \triangleq \begin{bmatrix} \theta_i \\ x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \theta_i \\ \boldsymbol{p}_i \end{bmatrix}, \tag{1}$$

where ω_i and v_i are, respectively, the angular velocity of the ith segment and longitudinal velocity of point P_i (both expressed in the body frame). The tractor velocity vector $\boldsymbol{u}_0 = [\omega_0 \; v_0]^\top \in \mathbb{R}^2$ is the only vehicle control input being at a control designer disposal. The nSNT kinematics depend on two kinds of parameters: trailer lengths $L_i > 0$ and hitching offsets $L_{hi} \neq 0$, where $L_{hi} > 0$ if the hitching point is located *behind* the wheels axle of a preceding segment, while $L_{hi} < 0$ in the opposite case (see Fig. 1). Due to the obvious mechanical reasons we assume that $|L_{hi}| < L_i$ if $L_{hi} < 0$. Due to interconnections between particular vehicle segments, their motion is not independent – velocity vectors of any two neighboring segments with kinematics (1) are related to each other by the formula

$$\boldsymbol{u}_i = \begin{bmatrix} -\frac{L_{hi}}{L_i}\mathrm{c}\beta_i & \frac{1}{L_i}\mathrm{s}\beta_i \\ L_{hi}\,\mathrm{s}\beta_i & \mathrm{c}\beta_i \end{bmatrix} \boldsymbol{u}_{i-1} = \boldsymbol{J}_i(\beta_i)\boldsymbol{u}_{i-1}, \tag{2}$$

where transformation matrix $\boldsymbol{J}_i(\beta_i)$ is invertible for any β_i if only $L_{hi} \neq 0$ allowing us to write

$$\boldsymbol{u}_{i-1} = \boldsymbol{J}_i^{-1}(\beta_i)\boldsymbol{u}_i = \begin{bmatrix} \frac{-L_i}{L_{hi}}\mathrm{c}\beta_i & \frac{1}{L_{hi}}\mathrm{s}\beta_i \\ L_i\,\mathrm{s}\beta_i & \mathrm{c}\beta_i \end{bmatrix} \boldsymbol{u}_i. \tag{3}$$

By combining (1) with (2) one easily finds a relation joining the control input $\boldsymbol{u}_0(t)$ with posture $\boldsymbol{q}_i$, namely:

$$\boldsymbol{q}_i(t) = \boldsymbol{q}_i(0) + \int_0^t \boldsymbol{G}(\boldsymbol{q}_i(\xi))\boldsymbol{H}_i(\xi)\boldsymbol{u}_0(\xi)d\xi, \tag{4}$$

where $\boldsymbol{H}_i(\xi) := \boldsymbol{I} = \mathrm{diag}\{1, 1\} \in \mathbb{R}^{2\times 2}$ for $i = 0$, while $\boldsymbol{H}_i(\xi) := \prod_{j=i}^{1} \boldsymbol{J}_j(\beta_j(\xi))$ for $i = 1, \ldots, N$.

Since positions $\boldsymbol{p}_i$ of points P_i are restricted at any time instant by two holonomic constraints (see Fig. 1)

$$x_{i-1}(t) = x_i(t) + L_i \cos\theta_i(t) + L_{hi}\cos\theta_{i-1}(t), \tag{5}$$

$$y_{i-1}(t) = y_i(t) + L_i \sin\theta_i(t) + L_{hi}\sin\theta_{i-1}(t), \tag{6}$$

the configuration $\boldsymbol{q}$ of the overall N-trailer can be uniquely represented by a minimal number of $N+3$ components

$$\boldsymbol{q} \triangleq [\beta_1 \;\ldots\; \beta_N \;\theta_j \;x_j \;y_j]^\top = \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{q}_j \end{bmatrix} \in \mathbb{T}^N \times \mathbb{R}^3, \tag{7}$$

where $\boldsymbol{\beta}$ is the joint-angles vector, the ith joint angle

$$\beta_i = \theta_{i-1} - \theta_i, \tag{8}$$

whereas $\boldsymbol{q}_j = [\theta_j \; x_j \; y_j]^\top = [\theta_j \; \boldsymbol{p}_j^\top]^\top$ is a posture of any selected vehicle segment for $j \in \{1, \ldots, N\}$. As a consequence, (7) defines the minimal set of measurements which shall be available in practice to uniquely determine position, orientation and *shape* of the vehicle chain in a task space. A kinematic model of the N-trailer can be expressed in a compact form of the driftless system (cf. [11], [13])

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{q}}_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_\beta(\boldsymbol{\beta}) \\ \boldsymbol{S}_j(\boldsymbol{\beta}, \boldsymbol{q}_j) \end{bmatrix} \boldsymbol{u}_0, \tag{9}$$

where

$$\boldsymbol{S}_\beta = \begin{bmatrix} \boldsymbol{c}^\top \boldsymbol{\Gamma}_1(\beta_1) \\ \boldsymbol{c}^\top \boldsymbol{\Gamma}_2(\beta_2)\boldsymbol{J}_1(\beta_1) \\ \vdots \\ \boldsymbol{c}^\top \boldsymbol{\Gamma}_N(\beta_N)\boldsymbol{J}_{N-1}^1(\boldsymbol{\beta}) \end{bmatrix}, \quad \boldsymbol{S}_j = \begin{bmatrix} \boldsymbol{c}^\top \boldsymbol{J}_j^1(\boldsymbol{\beta}) \\ \boldsymbol{d}^\top \boldsymbol{J}_j^1(\boldsymbol{\beta})\mathrm{c}\theta_j \\ \boldsymbol{d}^\top \boldsymbol{J}_j^1(\boldsymbol{\beta})\mathrm{s}\theta_j \end{bmatrix},$$

and $\boldsymbol{J}_j^1(\boldsymbol{\beta}) \triangleq \boldsymbol{J}_j(\beta_j)\boldsymbol{J}_{j-1}(\beta_{j-1})\ldots\boldsymbol{J}_1(\beta_1)$, $\boldsymbol{\Gamma}_i(\beta_i) \triangleq \boldsymbol{I} - \boldsymbol{J}_i(\beta_i)$, $\boldsymbol{I} = \mathrm{diag}\{1, 1\} \in \mathbb{R}^{2\times 2}$, $\boldsymbol{c}^\top \triangleq [1 \; 0]$, $\boldsymbol{d}^\top \triangleq [0 \; 1]$.

B. Control problem formulation

Let us consider the unicycle-admissible reference path[1] $\boldsymbol{q}_d(s) = [\beta_{d1}(s) \ \ldots \ \beta_{dN}(s) \ \theta_d(s) \ \boldsymbol{p}_d^\top(s)]^\top$, where

$$\boldsymbol{p}_d^\top(s) = \left[x_d(s) \ y_d(s)\right] \in \mathbb{R}^2 \tag{10}$$

is parametrized by the curvilinear path-length $s \in [0, \infty)$. Define the two-dimensional ball $B(\boldsymbol{p}_d(s), \varepsilon)$ of radius $\varepsilon \geq 0$ centered at point $\boldsymbol{p}_d(s)$, and the boundary set

$$B_\varepsilon \triangleq \bigcup_{s\in[0,\infty)} B(\boldsymbol{p}_d(s), \varepsilon) \subset \mathbb{R}^2.$$

Geometrical interpretation of the boundary set has been illustrated in Fig. 2, where B_ε determines the 2ε-wide symmetrical envelope around the reference path. The envelope

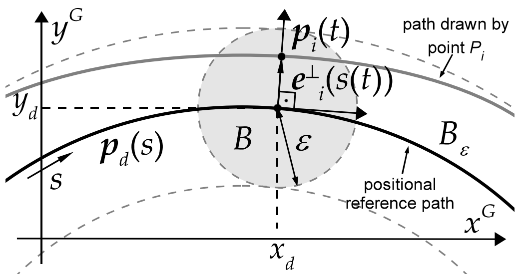

Fig. 2. Illustration of the positional reference path $\boldsymbol{p}_d(s)$, path $\boldsymbol{p}_i$ drawn by the ith characteristic point P_i, and definitions of the boundary set B_ε and orthogonal off-track error $\boldsymbol{e}_i^\perp(s(t))$

will represent the minimal free space required for the N-trailer to safely maneuver along the reference path. Thus, ε will correspond to the *boundary off-track* of the N-trailer

$$\varepsilon \triangleq \max_{i\in\{0,\ldots,N\}} \left\{ \sup_{t\geq T} \left\| \boldsymbol{e}_i^\perp(s(t)) \right\| \right\}, \tag{11}$$

where $\boldsymbol{e}_i^\perp(s(t)) \triangleq \boldsymbol{p}_i(t) - \boldsymbol{p}_d(s(t))$ is the orthogonal off-track error determined for point P_i of the vehicle along a reference path (see Fig. 2). Let $T = T(\varepsilon, \boldsymbol{q}(0))$ be a convergence time to set B_ε.

Definition 1 (Control problem): Design a feedback control law $\boldsymbol{u}_0(\boldsymbol{q}, \boldsymbol{q}_d, \cdot)$ which applied into (9) ensures boundedness $\| \boldsymbol{q}(t)\| < \infty$ for all $t \geq 0$, and the existence of a time instant $T(\varepsilon, \boldsymbol{q}(0)) \in [0, \infty)$ such that $\forall \, t \geq T$ hold:

(I) $v_i(t) > 0$ for $i = 0, \ldots, N$,

(II) $\boldsymbol{p}_i(t) \overset{(4)}{=} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_i(t) \in B_\varepsilon$ for all $i = 0, \ldots, N$ with the minimal boundary off-track (11).

Condition (I) imposes the forward motion strategy for all the vehicle segments. Condition (II) makes the above problem qualitatively different from the classical path-following task. In the latter case, one usually expects asymptotic following of the reference path by a single characteristic point – the so-called *guidance point* of a vehicle, usually selected at P_0 or P_N. Furthermore, in the classical approach time evolution of remaining characteristic points (except the guidance one) is secondary if only boundedness of $\boldsymbol{q}(t)$ is satisfied. In contrast, the control problem formulated in Definition 1 does not require (however does not exclude) exact reproduction of the path by any characteristic point of a vehicle, but it is rather expected that *all* the characteristic points of a vehicle (approximately) follow the reference path by staying inside the boundary set of a minimal width. Hence, the control problem stated above includes the classical path-following problem as a special case. Only in two particular cases – for rectilinear paths and for circular paths if $L_i = L_{hi}$ – the classical and the above stated path-following problems coincide leading to the perfect (asymptotic) reproduction of the reference path by all the characteristic points of a vehicle (the boundary off-track $\varepsilon = 0$).

[1]The *unicycle-admissibility* of the reference path means that vector $[\theta_d(s) \ x_d(s) \ y_d(s)]^\top$ satisfies unicycle kinematics for all $s \geq 0$.

III. Control Strategy Description

A. The reference path and the virtual guidance point

To make adoption of the control strategy proposed in [13] possible (cf. also [14]) let us assume that the positional reference path, in contrast to the parametrized form (10), can be determined as a set of pairs (x_d, y_d) satisfying equation

$$F(x_d, y_d) \triangleq \sigma f(x_d, y_d) = 0, \quad \sigma \in \mathbb{R} \setminus \{0\}, \tag{12}$$

where σ is a design parameter, the sign of which determines desired motion direction along the path, cf. [13]. Following [14] and [13], we assume that $F(x, y)$ is well defined for $(x, y) \in \mathcal{D} \subset \mathbb{R}^2$, that is: $F(x, y)$ is bounded and at least twice differentiable guaranteeing existence of derivatives $F_x(x, y) \triangleq \partial F(x, y)/\partial x$, $F_y(x, y) \triangleq \partial F(x, y)/\partial y$, $F_{z_1 z_2}(x, y) \triangleq \partial^2 F(x, y)/\partial z_1 \partial z_2$, $z_1, z_2 \in \{x, y\}$, while gradient $\nabla F(x, y) = \left[F_x(x, y) \ F_y(x, y)\right]$ is such that $\| \nabla F(x, y)\| > 0$ for $(x, y) \in \mathcal{D}$.

Let us introduce the *virtual guidance point* of a vehicle

$$\bar{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{w}) = \begin{bmatrix} \bar{\theta}(\boldsymbol{q}, \boldsymbol{w}) \\ \bar{x}(\boldsymbol{q}, \boldsymbol{w}) \\ \bar{y}(\boldsymbol{q}, \boldsymbol{w}) \end{bmatrix} \triangleq \sum_{i=0}^{N} w_i \boldsymbol{q}_i \tag{13}$$

being the weighted linear combination of postures $\boldsymbol{q}_i$ of all the vehicle segments (see (1)), where the weights in vector $\boldsymbol{w} = [w_0 \ \ldots \ w_N]^\top$ are such that

$$w_0 + w_1 + \ldots + w_N = 1. \tag{14}$$

The virtual guidance point (13) can be considered as a (weighted) averaged posture of the N-trailer vehicle; it will be used in the next section to the control law derivation.

Having the reference path defined by equation (12), the values of $F(\boldsymbol{q}_i) = \sigma f(x_i, y_i)$ and $F(\bar{\boldsymbol{q}}) = \sigma f(\bar{x}, \bar{y})$ can be treated as *signed distances* from, respectively, the ith vehicle segment and the virtual guidance point of a vehicle to the reference path[2] (note that by definition (12), $F(\boldsymbol{q}_i) = 0$ and $F(\bar{\boldsymbol{q}}) = 0$ only if points (x_i, y_i) and $(\bar{x}, \bar{y})$, respectively, are exactly on the reference path). Furthermore, if mapping F is such that locally around zero $\left\| \boldsymbol{e}_i^\perp \right\| \leq \kappa(|F(\boldsymbol{q}_i)|) \, \forall \, i$, where $\kappa(\cdot)$ is a positive definite and strictly increasing function, then the boundary off-track (11) locally satisfies

$$\varepsilon \leq \max_{i\in\{0,\ldots,N\}} \left\{ \sup_{t\geq T} \kappa(|F(\boldsymbol{q}_i(t))|) \right\}. \tag{15}$$

[2]However, $F(\cdot)$ is not equivalent to the Euclidean distance [14], [19].

Thus, minimization of the boundary off-track defined by (11) can be replaced by minimization of all the signed distances $F(\boldsymbol{q}_i(t))$ for $i = 0, \ldots, N$ and for all $t \geq T$.

B. Derivation of the cascade-like controller

For the purpose of further analysis let us define the *guidance error* as a function of the virtual guidance point

$$\boldsymbol{y}(\bar{\boldsymbol{q}}) = \begin{bmatrix} y_1(\bar{\boldsymbol{q}}) \\ y_2(\bar{\boldsymbol{q}}) \end{bmatrix} \triangleq \begin{bmatrix} \tilde{\theta}(\bar{\boldsymbol{q}}) \\ F(\bar{\boldsymbol{q}}) \end{bmatrix} \triangleq \begin{bmatrix} \bar{\theta} - \theta_d(\bar{x}, \bar{y}) \\ \sigma f(\bar{x}, \bar{y}) \end{bmatrix}, \qquad (16)$$

where $\theta_d(\bar{x}, \bar{y}) \triangleq \text{Atan2}\left(-F_x(\bar{x}, \bar{y}), F_y(\bar{x}, \bar{y})\right) \in (-\pi, \pi]$ is a tangency-angle to the level curve $F(\bar{x}, \bar{y}) = c$, while $\theta_d(x_d, y_d)$ becomes a reference orientation along (12).

In the first step of a control law derivation we assume that time-evolution of the virtual guidance point can be (approximately) governed by unicycle-like kinematics, i.e.

$$\dot{\bar{\boldsymbol{q}}} \triangleq \boldsymbol{G}(\bar{\boldsymbol{q}})\bar{\boldsymbol{u}}, \quad \boldsymbol{G}(\bar{\boldsymbol{q}}) \overset{(1)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\bar{\theta} & \sin\bar{\theta} \end{bmatrix}^\top \qquad (17)$$

with virtual control input $\bar{\boldsymbol{u}} = [\bar{u}_1 \ \bar{u}_2]^\top \in \mathbb{R}^2$. The main control concept is based on the postulate that the guidance error (16) can be made bounded and convergent to some neighborhood of $\boldsymbol{y}_{\lim} = [2\mu\pi \ 0]^\top$, $\mu \in \{0, \pm 1, \ldots\}$ by forcing

$$\bar{\boldsymbol{u}} = \boldsymbol{\Phi}(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d) = \begin{bmatrix} \Phi_\omega(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d) \\ \Phi_v(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d) \end{bmatrix} \in \mathbb{R}^2, \qquad (18)$$

where $\boldsymbol{\Phi}(\cdot)$ is the feedback control function originally proposed for unicycle kinematics in [19] and [14], and then adopted to the N-trailers in [13]. In the case of forward motion strategy, the feedback control function takes the form (cf. [19] and [13]):

$$\boldsymbol{\Phi} \triangleq \begin{bmatrix} -k\left(\frac{v_d \,\|\nabla F(\bar{\boldsymbol{q}})\| F(\bar{\boldsymbol{q}})}{\sqrt{1+F^2(\bar{\boldsymbol{q}})}} + \dot{F}(\bar{\boldsymbol{q}})\right) + \dot{\theta}_d(\bar{\boldsymbol{q}}) \\ v_d \end{bmatrix}, \qquad (19)$$

where $k > 0$ is a design parameter, $v_d = \text{const} > 0$ is a desired longitudinal velocity prescribed for the virtual guidance point, whereas

$$\begin{aligned}
\dot{F}(\bar{\boldsymbol{q}}) &= v_d\left[F_x(\bar{x}, \bar{y})\mathrm{c}\bar{\theta} + F_y(\bar{x}, \bar{y})\mathrm{s}\bar{\theta}\right], \\
\dot{\theta}_d(\bar{\boldsymbol{q}}) &= v_d \frac{F_1(\bar{\boldsymbol{q}})\mathrm{c}\bar{\theta} + F_2(\bar{\boldsymbol{q}})\mathrm{s}\bar{\theta}}{\|\nabla F(\bar{\boldsymbol{q}})\|^2}, \\
F_1(\bar{\boldsymbol{q}}) &= F_x(\bar{x}, \bar{y})F_{xy}(\bar{x}, \bar{y}) - F_y(\bar{x}, \bar{y})F_{xx}(\bar{x}, \bar{y}), \\
F_2(\bar{\boldsymbol{q}}) &= F_x(\bar{x}, \bar{y})F_{yy}(\bar{x}, \bar{y}) - F_y(\bar{x}, \bar{y})F_{xy}(\bar{x}, \bar{y}), \\
\|\nabla F(\bar{\boldsymbol{q}})\|^2 &= F_x^2(\bar{x}, \bar{y}) + F_y^2(\bar{x}, \bar{y}).
\end{aligned}$$

Now, the objective is to find how to satisfy postulate (18) by appropriate definition of the tractor control input $\boldsymbol{u}_0$. To this aim, one can utilize definition (13) with equations (1) and (2) to show that

$$\begin{aligned}
\dot{\bar{\boldsymbol{q}}} &\overset{(13)}{=} \sum_{i=0}^{N} w_i \dot{\boldsymbol{q}}_i \overset{(1)}{=} \sum_{i=0}^{N} w_i \boldsymbol{G}(\boldsymbol{q}_i)\boldsymbol{u}_i \\
&\overset{(2)}{=} \left[w_0 \boldsymbol{G}(\boldsymbol{q}_0) + \sum_{i=1}^{N} w_i \boldsymbol{G}(\boldsymbol{q}_i) \prod_{j=i}^{1} \boldsymbol{J}_j(\beta_j)\right] \boldsymbol{u}_0 \qquad (20) \\
&= \boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{u}_0, \qquad (21)
\end{aligned}$$

where matrix $\boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{w}) \in \mathbb{R}^{3\times 2}$ depends only on weights $\boldsymbol{w} = [w_0 \ \ldots \ w_N]^\top$ and on measurable configuration (7) due to the existence of geometric constraints (5), (6), and (8). Equating the right-hand sides of (21) and (17) with utilization of postulate (18) yields

$$\boldsymbol{G}(\bar{\boldsymbol{q}})\boldsymbol{\Phi}(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d) = \boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{u}_0.$$

Now, assuming that $\boldsymbol{\Gamma}(\boldsymbol{q}(t), \boldsymbol{w})$ has a full rank for all $t \geq 0$, solution of the above equation with respect to $\boldsymbol{u}_0$ yields the control law

$$\boldsymbol{u}_0(\boldsymbol{q}, \boldsymbol{w}, v_d) \triangleq \boldsymbol{\Gamma}^\dagger(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{G}(\bar{\boldsymbol{q}})\boldsymbol{\Phi}(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d), \qquad (22)$$

where $\boldsymbol{\Gamma}^\dagger = (\boldsymbol{\Gamma}^\top\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}^\top$ is the left pseudo-inverse of $\boldsymbol{\Gamma}$. Worth noting that control law (22) has a cascade-like structure illustrated by the block scheme in Fig. 3 with the outer-loop feedback controller $\boldsymbol{\Phi}(\boldsymbol{y}(\bar{\boldsymbol{q}}), v_d)$ and the inner-loop transformation represented by product $\boldsymbol{\Gamma}^\dagger(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{G}(\bar{\boldsymbol{q}})$. On the scheme, the terms F and v_d determine the reference path and reference velocity, respectively, while $\boldsymbol{w} = [w_0 \ \ldots \ w_N]^\top$ is a vector of user-defined weights which determine instantaneous location of the virtual guidance point of a vehicle. Let us comment three properties of controller (22) which will be denoted in the sequel by *P1-P3*.

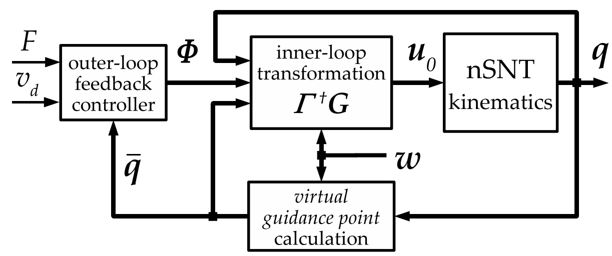

Fig. 3. Block scheme of the proposed cascade-like control system.

P1: It is not hard to check that when the virtual guidance point is particularly selected as $\bar{\boldsymbol{q}} = \boldsymbol{q}_i$, that is, if $w_i = 1$ and $w_j = 0$ for $j \neq i$, then assumption (17) is strictly satisfied and control law (22) takes the special form proposed in [13]:

$$\boldsymbol{u}_0(\boldsymbol{q}, v_d) = \prod_{j=1}^{i} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{\Phi}(\boldsymbol{y}(\boldsymbol{q}_i), v_d). \qquad (23)$$

Thus, (22) can be treated as a generalization of the control strategy presented in [13] for the case of forward motion. The main purpose of the generalization is to provide ability for more flexible selection of weights $\boldsymbol{w} = [w_0 \ \ldots \ w_N]^\top$ (apart from the special set mentioned above) in order to minimize the boundary off-track of a vehicle controlled by (22).

P2: Upon the main result presented in [13], one can state that application of control law (22) in its special form of (23) into kinematics (9) with sign-homogeneous offsets $L_{hj} < 0$ for $j = 1, \ldots, i$ guarantees asymptotic convergence of guidance error (16) to the point $\boldsymbol{y}_{\lim} = [2\mu\pi \ 0]^\top$ for all initial conditions $\boldsymbol{y}(\bar{\boldsymbol{q}}(0))$ with $(\bar{x}(0), \bar{y}(0)) \in \mathcal{D}$ and outside the set $\{\tilde{\theta} = (2\mu + 1)\pi, F(\bar{\boldsymbol{q}}) = 0\}$. Furthermore, convergence of the guidance error entails local asymptotic stability of joint-angle errors $\tilde{\beta}_j = \beta_{dj} - \beta_j$, $j = 1, \ldots, i$, for the so-called segment-platooning (S-P) reference paths[3]. One may expect that the rest of joint angles β_j for $j = i+1, \ldots, N$ shall stay bounded thanks to structural stability of joint-angles dynamics in (9) for the forward motion strategy (that is for $v_d > 0$). Although in this case the virtual guidance point $\bar{\boldsymbol{q}} = \boldsymbol{q}_i$ can asymptotically follow the reference path accurately, the boundary off-track of a vehicle can be excessively large. Since, in essence, the asymptotic convergence of guidance error (16) is not the control objective here, the more general selections of weights $w_0, \ldots, w_N$ may be beneficial in reducing the boundary off-track. Following the result recalled above, one may expect that also for more general sets of weights, for which assumption (17) may be satisfied only approximately, the guidance error (16) will be bounded and will approximately follow the reference path, leading however to smaller boundary off-track (11) than in the asymptotic case.

P3: Based on the stability analysis provided in [13], it can be shown that the nSNT kinematics equipped with positive hitching offsets and controlled with (22) may exhibit the so-called jackknife phenomenon in the forward motion strategy (as a consequence of the non-minimum-phase closed-loop dynamics, see [11]). To avoid the jackknifing behavior of a vehicle, the following change in calculation of matrix $\boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{w})$ is proposed. Namely, for the joints with $L_{hj} > 0$ it is proposed to take in the matrix product of (20) the modified transformation matrix (instead of the one defined by (2)):

$$\hat{\boldsymbol{J}}_j(\beta_j) \triangleq \begin{bmatrix} -\frac{\rho_j}{L_j}\mathrm{c}\beta_j & \frac{1}{L_j}\mathrm{s}\beta_j \\ \rho_j\,\mathrm{s}\beta_j & \mathrm{c}\beta_j \end{bmatrix} \quad \text{with} \quad \rho_j \triangleq -L_{hj}. \quad (24)$$

Clearly, matrix (24) utilizes the hitching offset with the opposite sign. Numerous simulation trials have shown that usage of this simple modification allows avoiding the jackknife effect, still letting one to minimize the boundary off-track of a vehicle (see Section IV).

C. Selection of weights $w_0, \ldots, w_N$

Values of weights $\boldsymbol{w} = [w_0 \ \ldots \ w_N]^\top$ can be flexibly chosen by a designer taking into account constraint (14). One assumes that the weights used in control law (22) are constant, thus they can be selected off-line before application of the controller. The most common way of selection is the trials-and-errors method. In this approach, the first step is to check the obtainable boundary off-track for particular sets of weights mentioned in the comment of property *P1*, sequentially for $i = 0$ to $i = N$. However, in most cases the more general selection is beneficial leading to smaller boundary off-track. When the number N is large, searching for the set of weights by trials-end-errors may be burdening or time-consuming. In this case, one may employ the automated numerical optimization search, for instance the extremum-seeking approach (see [5]) to minimize an appropriately constructed quality criterion.

[3]The reference path is called S-P if all the reference vehicle segments have non-zero longitudinal velocities of the same sign along the path [13].

IV. Simulation Examples

Effectiveness of the proposed control strategy has been illustrated by two sets of simulations: for the circular reference path defined by equation $x_d^2 + y_d^2 - R_d^2 = 0$, $R_d = 1.5\,\mathrm{m}$, and for the sine-type reference path represented by equation $y_d - 1.5\sin(0.8x_d) = 0$. The vehicle kinematic parameters and control design parameters have been chosen as follows: $L_1 = 0.7\,\mathrm{m}$, $L_{h1} = -0.1\,\mathrm{m}$, $L_2 = L_3 = 0.6\,\mathrm{m}$, $L_{h2} = L_{h3} = +0.1\,\mathrm{m}$, $k = 2$, $v_d = 1.5\,\mathrm{m/s}$, $\sigma = +1$. Seven tests were performed for the circular path with different sets of weights, S1 to S7, collected in Table I. The table includes values of boundary off-track ε and bias b obtained under steady motion conditions for particular sets of weights, which have been computed as $\varepsilon = \max\{|R_d - R_M|, |R_d - R_m|\}$, $b = 0.5(R_M + R_m) - R_d$, where R_M and R_m denote, respectively, the maximal and minimal radius of circular paths drawn by the vehicle segments during steady motion.

TABLE I

Obtained boundary off-track and bias for the circular path

set of selected weights	ε [m]	b [m]
S1: $w_0 = 1$, $w_1 = 0$, $w_2 = 0$, $w_3 = 0$	0.466	-0.233
S2: $w_0 = 0$, $w_1 = 1$, $w_2 = 0$, $w_3 = 0$	0.255	-0.051
S3: $w_0 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0$	0.310	$+0.128$
S4: $w_0 = 0$, $w_1 = 0$, $w_2 = 0$, $w_3 = 1$	0.413	$+0.244$
S5: $w_0 = 0.44$, $w_1 = 0.31$, $w_2 = 0.25$, $w_3 = 0$	0.202	-0.005
S6: $w_0 = w_1 = w_2 = w_3 = 0.25$	0.349	$+0.173$
S7: $w_0 = 0$, $w_1 = w_2 = 0.5$, $w_3 = 0$	0.262	$+0.075$

Worth emphasizing that two sets of symmetric weights, S6 and S7, did not led to a minimal boundary off-track. The best results were obtained for asymmetric weights from set S5, which has been found based on the output of the extremum-seeking optimization procedure employed for quality criterion $V \triangleq (\sum_{j=0}^{3} F(\boldsymbol{q}_j))^2$ motivated by relation (15). Two exemplary plots of vehicle motion for sets S1 and S5 are presented in Fig.4 (A) and (B), respectively. The plots in Fig.4 (C) show a control performance for the sine-type (curvature-varying) reference path employing the same set S5 of optimized weights. The symmetrical $\pm 0.25\,\mathrm{m}$ envelopes have been denoted by dashed red lines around the reference paths to facilitate assessment of the obtained control performance in a task space. Example (C) shows that the optimal weights computed for the circular path can be employed also to other types of paths preserving comparable control performance.

V. Conclusions

The presented control strategy is an alternative solution to the problem of path-following with reduced off-track addressed in [3] and [4]. Worth stressing high scalability of the new control approach with respect to a number of trailers and its utility value, where one may easily influence a boundary off-track of the nSNT vehicle by only changing the weights in a definition of the virtual guidance point. An interesting topic for further research may come from letting the weights to be time-varying making the overall control algorithm adaptive with respect to changes in the instantaneous curvature of a reference path.

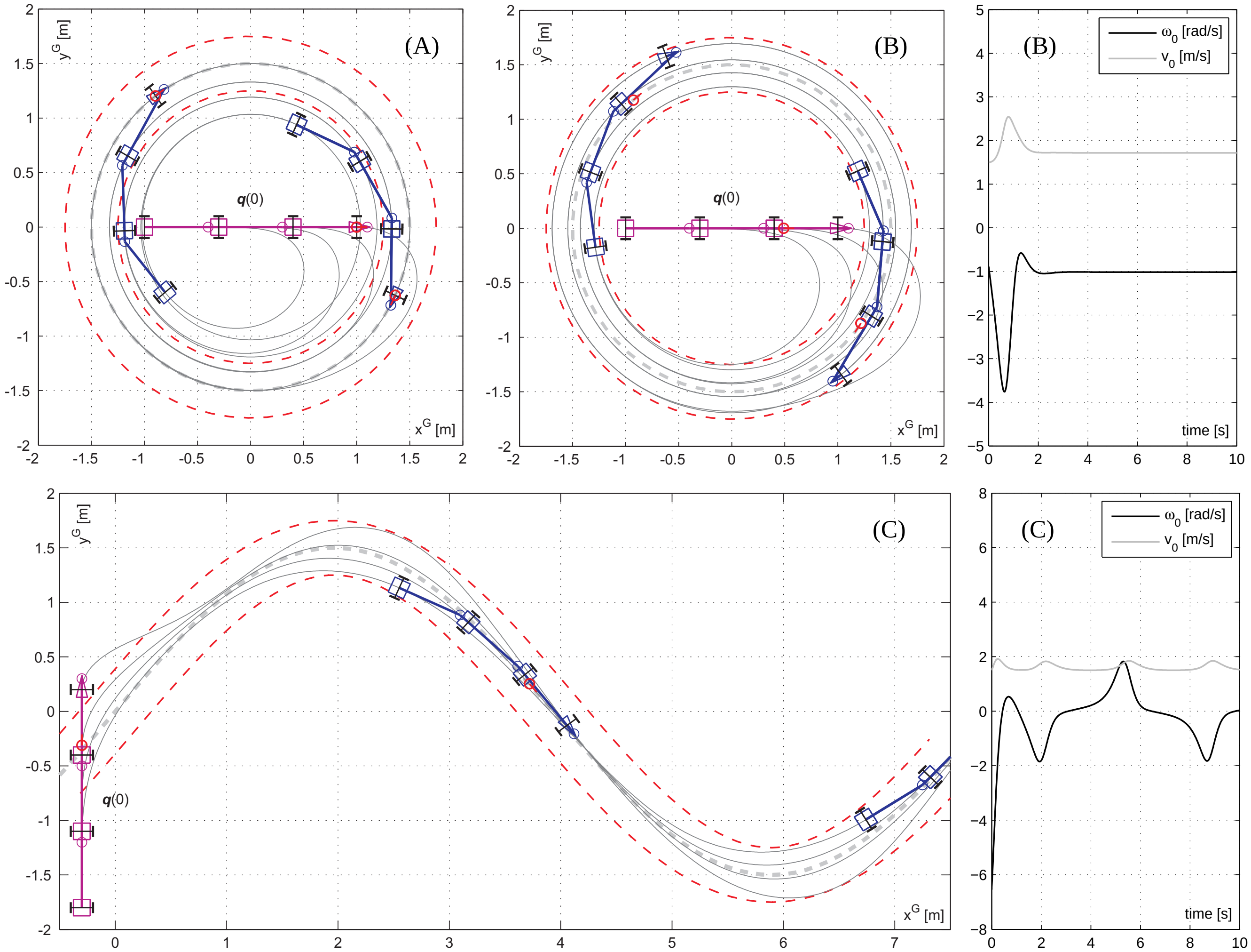

Fig. 4. Simulation results for two different reference paths: for the circular path with weights from set S1 (A), for the circular path with weights from optimized set S5 (B), and for the sine-type path with weights from optimized set S5 (C); reference paths have been denoted by dashed grey lines, initial vehicle configuration $\boldsymbol{q}(0)$ has been highlighted in magenta, while the virtual guidance point has been denoted by the red circular mark.

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