Cascaded approach to the path-following problem for N-trailer robots

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***Abstract*— By application of the cascaded control paradigm a highly-scalable control law is proposed for articulated mobile robots equipped with arbitrary number of trailers (N-trailers) for the path-following task. The concept relies on utilization of the novel path-following controller, recently published by Morro et al., devised for the unicycle kinematics of the last trailer (outer loop), and then on transformation of the resultant control functions along a vehicle kinematic chain to a tractor segment (inner loop). A control law proposed in the paper can be applied into all the known types of N-trailer robots (nSNT, GNT, and SNT) under the *sign-homogeneity* assumption for hitching offsets of trailers. The method has been examined by simulation examples.**

I. INTRODUCTION

From the control perspective, the N-trailer robotic vehicles (shortly: N-trailers) are the challenging and very interesting plants, therefore they still occupy attention of researchers and engineers. N-trailers consist of an active tractor (usually a differentially-driven unicycle-like vehicle) with arbitrary number of single-axle trailers interconnected in a chain by the passive rotary joints. In the literature one distinguishes three kinds of N-trailer robots: Standard N-Trailers (SNT) with all the joints of on-axle type [9], non-Standard N-Trailers (nSNT) with all the joints of off-axle type [7], and General N-Trailers (GNT) with mixed on-axle and off-axle hitched joints [1]. Models of N-trailers are characterized by several specific properties like high nonlinearity, substantially less number of control inputs in relation to a number of controlled variables, structural instability in backward motion, and the nonminimum-phase property in the presence of off-axle interconnections of trailers [8]. Due to the complicated nature of N-trailer kinematics some control problems still remain unsolved, while the other ones involve further investigations to make the control laws more practical and simpler in implementation.

The path-following task, which is considered in this work, belongs to the classical set of motion problems defined for mobile robots which is of a particular practical meaning and has been treated by several investigators with relation to articulated vehicles. Exemplary solutions of the task devised for the tractor-single-trailer systems can be found in [5], [13], [3]. Solutions applicable to vehicles with arbitrary number of trailers have been proposed in [4] (although presented only for 2-trailers), [2], [11], and [15].

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In this paper a highly-scalable cascaded control method is presented, which allows solving the path following task for N-trailer robots with the guidance point located on the last vehicle segment. Application of the cascaded control paradigm into N-trailers is not a completely new concept. It has been independently developed and recently presented for a backward pushing task in [10], for a trajectory-tracking task in [6], and for set-point control in [7]. A novelty of the solution proposed here comes from combination of two components. The first one is application of the cascaded approach to the path following task which admits backward and forward motion strategies for a vehicle. The second one results from utilization of a relatively new path-following concept developed for unicycle kinematics in [12]. The controller presented in [12] is especially interesting, since it treats the path-following problem in a completely new way which removes the main limitations of the well known and frequently utilized method of Samson [14]. The new cascaded solution will be derived for a kinematic model of nSNT robots treating velocities of a tractor segment as control inputs. After simple approximations of the inner loop, it will be shown how the method can be applied also into GNT and SNT kinematics. Justification of the pure kinematic approach comes from the fact that most difficulties with maneuvers performed by N-trailers arise just on the kinematic level.

II. VEHICLE KINEMATICS AND CONTROL PROBLEM

A. Kinematics of N-trailer robots

Kinematics of N-trailer robots will be described upon the scheme presented in Fig. 1. N-trailer robot consists of a differentially driven tractor (segment number 0) and N trailers interconnected in a chain by passive rotary joints. Kinematic parameters of the robot are: trailer lengths $L_i > 0$, and hitching offsets $L_{hi} \in \mathbb{R}$, $i = 1, \dots, N$. The i-th offset is positive when the i-th joint is located *behind* the wheels-axle of a preceding segment, while it is negative in the opposite case. One says that the i-th joint is of *on-axle* type if $L_{hi} = 0$, and of *off-axle* type when $L_{hi} \neq 0$. Configuration of the system can be uniquely determined by the vector

$$\boldsymbol{q} \triangleq [\beta_1 \ \dots \ \beta_N \ \theta_N \ x_N \ y_N]^\top = \left[\boldsymbol{\beta}^\top \ \boldsymbol{q}_N^\top\right]^\top, \qquad (1)$$

where $\boldsymbol{\beta} = [\beta_1 \ \dots \ \beta_N]^\top \in \mathbb{T}^N$ is a sub-vector of joint angles, while $\boldsymbol{q}_N = [\theta_N \ x_N \ y_N]^\top \in \mathbb{R}^3$ determines a posture of the last robot segment. Variables x_N and y_N denote coordinates of the *guidance point* P located at the mid-point of the last-trailer wheels-axle. The only control

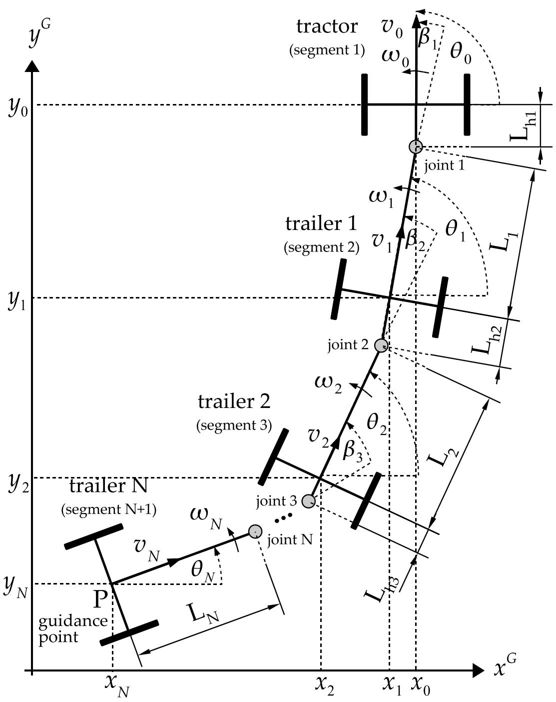

Fig. 1. Configuration and kinematic parameters of the N-trailer robot

inputs of the robot are the tractor velocities $\boldsymbol{u}_0 = [\omega_0 \ v_0]^\top$, where ω_0 and v_0 are the angular and longitudinal velocities, respectively.

Under the rolling-without-skidding assumption one can treat any i-th vehicle segment ($i = 1, \ldots, N$) as the unicycle[1]

$$\dot{\theta}_i = \omega_i, \quad \dot{x}_i = v_i \mathrm{c}\theta_i, \quad \dot{y}_i = v_i \mathrm{s}\theta_i \tag{2}$$

with virtual input $\boldsymbol{u}_i = [\omega_i \ v_i]^\top$ where ω_i and v_i are the angular and longitudinal velocities of the i-th segment, respectively. It can be shown that velocities of any two neighboring segments for $i = 1, \ldots, N$ are related by the formula

$$\boldsymbol{u}_i = \boldsymbol{J}_i(\beta_i)\boldsymbol{u}_{i-1}, \quad \boldsymbol{J}_i(\beta_i) = \begin{bmatrix} -\frac{L_{hi}\mathrm{c}\beta_i}{L_i} & \frac{\mathrm{s}\beta_i}{L_i} \\ L_{hi}\,\mathrm{s}\beta_i & \mathrm{c}\beta_i \end{bmatrix}, \tag{3}$$

where $\boldsymbol{J}_i(\beta_i)$ is the transformation matrix, which is invertible for $L_{hi} \neq 0$ (i.e. when the i-th joint is of off-axle type). By iterative application of (3) for $i = N, \ldots, 1$ one obtains the velocity propagation equation

$$\boldsymbol{u}_N = \boldsymbol{J}_N(\beta_N) \ldots \boldsymbol{J}_2(\beta_2)\boldsymbol{J}_1(\beta_1)\boldsymbol{u}_0 = \prod_{j=N}^{1} \boldsymbol{J}_j(\beta_j)\boldsymbol{u}_0, \tag{4}$$

which describes how the tractor control input $\boldsymbol{u}_0$ influences velocity $\boldsymbol{u}_N$ of the last trailer. In order to make the basic formulation of N-trailer kinematics complete one should recall equation valid for the joint angle: $\beta_i = \theta_{i-1} - \theta_i$.

B. Control problem formulation

To state the control problem precisely enough, let us follow the concept presented in [12] and define the reference path on the motion plane in the form of implicit equation

$$F(x, y) \triangleq \sigma f(x, y) = 0, \tag{5}$$

where $\sigma \in \{-1, +1\}$ is a decision factor which will allow selecting a desired quadrant for a reference orientation along the path. It is assumed that $f(x, y)$ is a scalar function at least twice differentiable with respect to the both arguments, meaning that the partial derivatives

$$F_x \triangleq \frac{\partial F(x,y)}{\partial x}, \quad F_y \triangleq \frac{\partial F(x,y)}{\partial y},$$
$$F_{xx} \triangleq \frac{\partial^2 F(x,y)}{\partial x^2}, \quad F_{yy} \triangleq \frac{\partial^2 F(x,y)}{\partial y^2}, \quad F_{xy} \triangleq \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

exist and are bounded in any bounded domain $D \subset \mathbb{R}^2$ where function F is bounded as well. Furthermore, it is assumed (cf. [12]) that the gradient

$$\nabla F(x, y) = \left[F_x(x, y) \ F_y(x, y)\right] \triangleq \left[\frac{\partial F(x,y)}{\partial x} \quad \frac{\partial F(x,y)}{\partial y}\right]$$

is such that $\|\nabla F(x, y)\| > 0$ for all $(x, y) \in D \subset \mathbb{R}^2$. The desired orientation, which determines direction tangent to path $F(x, y) = 0$, can be defined as

$$\theta_d(x, y) \triangleq \text{Atan2c}\left(-F_x(x, y), F_y(x, y)\right) \in \mathbb{R}, \tag{6}$$

which is always well determined for $\|\nabla F(x, y)\| > 0$, and where $\text{Atan2c}(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous version of the four-quadrant function $\text{Atan2}(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \mapsto (-\pi, \pi]$. Decision factor $\sigma \in \{-1, +1\}$, introduced in (5) and included in partial derivatives F_x and F_y, determines a desired quadrant for reference orientation $\theta_d(x, y)$. Now, by referring to the last-trailer kinematics in the form of (2), one can introduce the path-following error for the last trailer

$$\boldsymbol{e}(\boldsymbol{q}_N) \triangleq \begin{bmatrix} F(\bar{\boldsymbol{q}}_N) \\ e_\theta(\boldsymbol{q}_N) \end{bmatrix} \triangleq \begin{bmatrix} \sigma f(\bar{\boldsymbol{q}}_N) \\ \theta_N - \theta_d(\bar{\boldsymbol{q}}_N) \end{bmatrix} \in \mathbb{R}^2, \tag{7}$$

where $\bar{\boldsymbol{q}}_N = [x_N \ y_N]^\top$ is a position vector of the guidance point P (cf. Fig. 1), and $f(\bar{\boldsymbol{q}}_N) \equiv f(x_N, y_N)$ determines function $f(x, y)$ evaluated at point $\bar{\boldsymbol{q}}_N$. In definition (7) the first component $F(\bar{\boldsymbol{q}}_N)$ can be called the *signed distance value* (see [12]) between the guidance point P and the desired path (note that $F(\bar{\boldsymbol{q}}_N) = 0$ only when the last trailer is on the desired path), while the second component, $e_\theta(\boldsymbol{q}_N)$, is the orientation error where, according to (6), $\theta_d(\bar{\boldsymbol{q}}_N) = \text{Atan2c}\left(-F_x(\bar{\boldsymbol{q}}_N), F_y(\bar{\boldsymbol{q}}_N)\right)$.

The control problem under consideration is to find a bounded feedback control law $\boldsymbol{u}_0(\boldsymbol{\beta}, \boldsymbol{q}_N, t)$ which guarantees that error (7) is convergent in the sense:

$$\lim_{t\to\infty} F(\bar{\boldsymbol{q}}_N(t)) = 0, \quad \lim_{t\to\infty} e_\theta(\boldsymbol{q}_N(t)) = 2\nu\pi, \ \nu \in \mathbb{Z}. \tag{8}$$

In the sequel, a solution to the above problem will be provided by using the cascaded-control approach.

[1]Short notation will be used for compactness: $\mathrm{c}\alpha \equiv \cos\alpha$, $\mathrm{s}\alpha \equiv \sin\alpha$.

III. CASCADED CONTROL CONCEPT FOR N-TRAILERS

A. Derivation of a control structure for nSNT robots

Assume that all the hitching offsets L_{hi} in the vehicle are

A1: non-zero: $L_{hi} \neq 0$ for any $i = 1, \ldots, N$,

A2: sign-homogeneous: $\text{sgn}(L_{hi}) = \text{sgn}(L_{hj})$ for any $i \neq j$.

Assumption A1 restricts our considerations to nSNT vehicles, while A2 guarantees that the hitching offsets are all positive or all negative. Under assumption A1 one can determine an inverse transformation to (3) in the form

$$\boldsymbol{u}_{i-1} = \boldsymbol{J}_i^{-1}(\beta_i)\boldsymbol{u}_i, \quad \boldsymbol{J}_i^{-1}(\beta_i) = \begin{bmatrix} \frac{-L_i \text{c}\beta_i}{L_{hi}} & \frac{\text{s}\beta_i}{L_{hi}} \\ L_i \, \text{s}\beta_i & \text{c}\beta_i \end{bmatrix}, \quad (9)$$

where $\boldsymbol{J}_i^{-1}(\beta_i)$ is the inverse-transformation matrix. By combination of formula (9) for all $i = 1, \ldots, N$ one obtains the following velocity propagation equation

$$\boldsymbol{u}_0(\boldsymbol{\beta}) = \prod_{j=1}^{N} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{u}_N, \quad (10)$$

which determines how the velocity vector $\boldsymbol{u}_N = [\omega_N \ v_N]^\top$ may be forced by the tractor input $\boldsymbol{u}_0$. Since (10) is an algebraic relation, velocities of the last trailer can be forced by the tractor inputs in the instantaneous manner as a function of the current joint-angles only. Eq. (10) defines in fact a closed-loop subsystem with feedback from angles $\boldsymbol{\beta}$. It will be shown that propagation formula (10) can be treated as an inner loop of a resultant control cascade with input $\boldsymbol{u}_N$ and output $\boldsymbol{u}_0$.

Now, let us treat the last trailer as the unicycle with virtual control input $\boldsymbol{u}_N = [\omega_N \ v_N]^\top$. Suppose that one would like to force $\boldsymbol{u}_N = \boldsymbol{\Phi}(\boldsymbol{q}_N, t)$ with $\boldsymbol{\Phi}(\boldsymbol{q}_N, t) \in \mathbb{R}^2$ being some feedback control function which ensures guidance of the last trailer toward (and along) a reference path (any particular form of function $\boldsymbol{\Phi}$ is not considered now – it will be given in Section III-C). According to (10) the control strategy for the tractor segment can take the following form

$$\boldsymbol{u}_0(\boldsymbol{\beta}, \boldsymbol{\Phi}(\boldsymbol{q}_N, t)) \triangleq \prod_{j=1}^{N} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{\Phi}(\boldsymbol{q}_N, t). \quad (11)$$

One may check that application of control law (11) into kinematics of nSNT robot gives:

$$\boldsymbol{u}_N \overset{(4)}{=} \prod_{j=N}^{1} \boldsymbol{J}_j(\beta_j)\boldsymbol{u}_0 \overset{(11)}{=} \prod_{j=N}^{1} \boldsymbol{J}_j(\beta_j) \prod_{j=1}^{N} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{\Phi}(\boldsymbol{q}_N, t) = \boldsymbol{\Phi}(\boldsymbol{q}_N, t). \quad (12)$$

Hence, application of control law (11) ensures that the last trailer moves in a way as it would be directly controlled by function $\boldsymbol{\Phi}$. In this context, one can treat (11) as a cascade connection of the inner-loop controller defined by (10) with the outer-loop controller defined by feedback function $\boldsymbol{\Phi}(\boldsymbol{q}_N, t)$. Assuming that the outer-loop control function is bounded in the sense that $\forall\, \boldsymbol{q}_N, t \ \|\boldsymbol{\Phi}(\boldsymbol{q}_N, t)\| \leq \phi_{\max} < \infty$, one can easily check boundedness of control law (11):

$$\|\boldsymbol{u}_0(\boldsymbol{\beta}, \boldsymbol{\Phi})\| \overset{(11)}{=} \left\| \prod_{j=1}^{N} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{\Phi}(\boldsymbol{q}_N, t) \right\| \leq \prod_{j=1}^{N} \|\boldsymbol{J}_j^{-1}(\beta_j)\| \, \phi_{\max} = \prod_{j=1}^{N} M_j \phi_{\max}, \quad (13)$$

where $M_j = \sqrt{\left(1 + \frac{L_j^2}{L_{hj}^2}\right) \text{c}^2\beta_j + \left(\frac{1}{L_{hj}^2} + L_j^2\right) \text{s}^2\beta_j}$ is the Frobenius norm of matrix $\boldsymbol{J}_j^{-1}(\beta_j)$, which is bounded under assumption A1.

Summarizing, the proposed control law lies in the cascade combination of two feedback loops: the outer loop defined by control function $\boldsymbol{\Phi}(\boldsymbol{q}_N, t)$, which is responsible for providing the instantaneous guiding-velocities for the last trailer upon a current path-following error, and the inner loop (defined by transformation (11)) which recomputes these guiding-velocities into the instantaneous control inputs for the tractor.

Before defining the outer-loop function $\boldsymbol{\Phi}(\boldsymbol{q}_N, t)$ in its explicit form, let us explain how one can extend application of controller (11) into N-trailers comprising on-axle joints.

B. Approximated inner loop for GNT and SNT robots

In case when any i-th joint is of on-axle type the hitching offset $L_{hi} = 0$ and the transformation matrix in (3) becomes singular (one cannot use the inverse relation (9)). However, one can propose to replace the transformation matrix for the on-axle joint by its non-singular approximation

$$\hat{\boldsymbol{J}}_i(\beta_i, \varepsilon_i) \triangleq \begin{bmatrix} -\frac{\varepsilon_i}{L_i}\text{c}\beta_i & \frac{1}{L_i}\text{s}\beta_i \\ \varepsilon_i \, \text{s}\beta_i & \text{c}\beta_i \end{bmatrix}, \quad \varepsilon_i \neq 0, \quad (14)$$

where ε_i is a prescribed sufficiently small non-zero parameter which satisfies the sign-homogeneity assumption (A2): $\text{sgn}(\varepsilon_i) = \text{sgn}(L_{hj})$ for any $j \neq i$ for which the joint is of off-axle type (mixed types of trailers hitching are characteristic for GNT vehicles). In case of SNT vehicles, where all the joints are of on-axle type, the common sign for all the parameters ε_i can be chosen arbitrarily. Now, one can inverse the approximate transformation matrix (14) obtaining

$$\hat{\boldsymbol{J}}_i^{-1}(\beta_i, \varepsilon_i) = \begin{bmatrix} -\frac{L_i}{\varepsilon_i}\text{c}\beta_i & \frac{1}{\varepsilon_i}\text{s}\beta_i \\ L_i \, \text{s}\beta_i & \text{c}\beta_i \end{bmatrix}, \quad (15)$$

which can be used in the cascade controller (11) in places corresponding to the on-axle joints. Since parameter ε_i resides in a denominator of particular elements in (15), it can be expected that a value of ε_i will influence sensitivity of a closed loop system to the presence of measurement noises in the inner and outer loops. Hence in practice, selection of ε_i should result from a compromise between precision of approximation (15) and the noise-sensitivity of a resultant closed-loop system.

C. Outer-loop controller

To complete a definition of the cascaded path-following controller for N-trailers it suffices to determine the outer-loop control function $\boldsymbol{\Phi}(\boldsymbol{q}_N, t)$. To this aim the novel control law presented in [12] will be utilized.

The implicit equation (5) defines the desired path which the last trailer (guidance segment) should follow. By referring to kinematics (2) for $i = N$, the path-following controller proposed in [12] can be described as follows:

$$\boldsymbol{\Phi}(\boldsymbol{q}_N, t) = \left[\Phi_\omega(\bar{\boldsymbol{q}}_N, t) \quad \Phi_v(t)\right]^\top \in \mathbb{R}^2, \qquad (16)$$

with

$$\Phi_\omega(\bar{\boldsymbol{q}}_N, t) \triangleq -k_1 \| \nabla F(\bar{\boldsymbol{q}}_N)\| \, \Phi_v(t) \frac{k_2 F(\bar{\boldsymbol{q}}_N)}{\sqrt{1+F^2(\bar{\boldsymbol{q}}_N)}} - k_1 |\Phi_v(t)| \nabla F(\bar{\boldsymbol{q}}_N) \cdot \bar{\boldsymbol{g}}_2(\theta_N) + \dot{\theta}_d, \qquad (17)$$

$$\Phi_v(t) \triangleq v_d(t), \qquad (18)$$

where $\nabla F(\bar{\boldsymbol{q}}_N) \equiv \nabla F(x_N, y_N)$ determines the gradient of function $F(\cdot)$ evaluated at current position $\bar{\boldsymbol{q}}_N = [x_N \; y_N]^\top$, (17) and (18) are, respectively, the angular and longitudinal control functions for the unicycle kinematics of the last trailer, $k_1 > 0$, $k_2 \in (0, 1]$ are the design parameters, and

$$\bar{\boldsymbol{g}}_2(\theta_N) = \left[\mathrm{c}\theta_N \;\; \mathrm{s}\theta_N\right]^\top, \qquad (19)$$

$$\dot{\theta}_d = \Phi_v(t) \frac{F_1(\bar{\boldsymbol{q}}_N)\mathrm{c}\theta_N + F_2(\bar{\boldsymbol{q}}_N)\mathrm{s}\theta_N}{\| \nabla F(\bar{\boldsymbol{q}}_N)\|^2}, \qquad (20)$$

$$F_1(\bar{\boldsymbol{q}}_N) = F_x(\bar{\boldsymbol{q}}_N)F_{xy}(\bar{\boldsymbol{q}}_N) - F_y(\bar{\boldsymbol{q}}_N)F_{xx}(\bar{\boldsymbol{q}}_N), \qquad (21)$$

$$F_2(\bar{\boldsymbol{q}}_N) = F_x(\bar{\boldsymbol{q}}_N)F_{yy}(\bar{\boldsymbol{q}}_N) - F_y(\bar{\boldsymbol{q}}_N)F_{xy}(\bar{\boldsymbol{q}}_N). \qquad (22)$$

The term $\dot{\theta}_d$ denotes the desired angular velocity, which is well determined along the curve $F(x, y) = 0$ under assumption $\| \nabla F(x, y)\| > 0$. Assume that longitudinal velocity (18) is defined by the bounded function $v_d(t)$ such that $\dot{v}_d(t)$ is bounded and $\lim_{t\to\infty} v_d(t) \neq 0$. According to Theorem 2 and the proof presented in [12] one should expect that direct application of (17)-(18) into unicycle kinematics (2) guarantees (under assumptions mentioned above) asymptotic convergence of error (7) in the sense determined by (8) for any initial condition $\boldsymbol{e}(\boldsymbol{q}_N(0))$, $\bar{\boldsymbol{q}}_N(0) \in D$, outside the set of unstable equilibria: $\{F(\bar{\boldsymbol{q}}_N) = 0, e_\theta = (2\nu+1)\pi) | \nu \in \mathbb{Z}\}$.

According to (12) one concludes that application of the inner-loop controller (11) with the outer-loop control function (17)-(18) into nSNT kinematics guarantees preservation of the asymptotic convergence result for the last trailer under the same conditions. On the other hand, Eq. (12) does not hold for approximation (15), thus generally one should not expect asymptotic convergence of error (7) if approximation (15) is used in the inner loop (cf. Section IV-B).

Worth to note that the path-following controller defined by control functions (17)-(18) does not involve determination of an instantaneous shortest distance to the path, which is required by the classical path-following control approach proposed in [14]. While the unique determination of the shortest distance usually imposes very conservative restrictions on the admissible curvature of the path and may be problematic in practice, the alternative solution presented in [12] is free of this limitations. The main restriction of the new method results from the fact that the desired path must be expressed analytically by implicit equation (5). Thus the algorithm does not cope directly with the non-analytical paths (cf. [12]).

The resultant cascaded control system for the N-trailer robots can be summarized by the block scheme presented in Fig. 2, where the inner-loop controller and the outer-loop controller have been defined by equations (11) and (16), respectively.

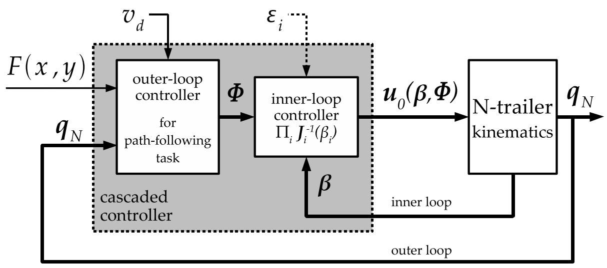

Fig. 2. Cascaded control system proposed for N-trailer robots (parameters ε_i are used only for GNT and SNT robots)

IV. Numerical examples

A. Simulation conditions

Effectiveness of the proposed control strategy has been examined by numerical simulations conducted for:

- elliptical path defined by function

$$f(x, y) := \frac{x^2}{A^2} + \frac{y^2}{B^2} - 1, \quad A = 2, B = 1, \qquad (23)$$

- S-like path defined by function

$$f(x, y) := y - B \tanh(Ax), \quad A = 5, B = 1. \qquad (24)$$

Four sets of simulation results, denoted by Sim1 to Sim4, have been collected for three-trailer robot kinematics ($N = 3$) with trailers of lengths $L_i = 0.3\,\mathrm{m}$, $i = 1, 2, 3$. In all simulations the following common values of control parameters have been used: $k_1 = 2$ and $k_2 = 1$. The results have been presented in Figs. 3-6 and they are commented on in Section IV-B. Apart from the tractor inputs (ω_0 and v_0) the control functions Φ_ω and Φ_v have been shown as well. On the X-Y plots the reference path has been depicted by the dashed gray line. The last trailer has been highlighted in red, while the initial robot configuration $\boldsymbol{q}_0$ has been highlighted in magenta.

a) Simulations for nSNT robots: Examples Sim1 and Sim2 provide the results for the nominal case where assumptions A1 and A2 are satisfied. nS3T kinematics of a three-trailer robot has been used with all the non-zero sign-homogeneous hitching offsets. In simulation Sim1 the following parameters and initial conditions have been selected: $L_{hi} = 0.1\,\mathrm{m}$ ($i = 1, 2, 3$), $v_d = -0.3$ m/s (backward motion strategy prescribed), $\sigma = +1$, and $\boldsymbol{q}_0 = [0 \;\; -1 \;\; 0]^\top$. The results of following the elliptical reference path (23) are presented in Fig. 3. For simulation Sim2 the following values have been chosen: $L_{hi} = -0.05\,\mathrm{m}$ ($i = 1, 2, 3$), $v_d = 0.15$ m/s (forward motion strategy prescribed), $\sigma = +1$, and $\boldsymbol{q}_0 = [0 \; -2 \; -0.8]^\top$. The results of following the S-like reference path (24) are provided in Fig. 4.

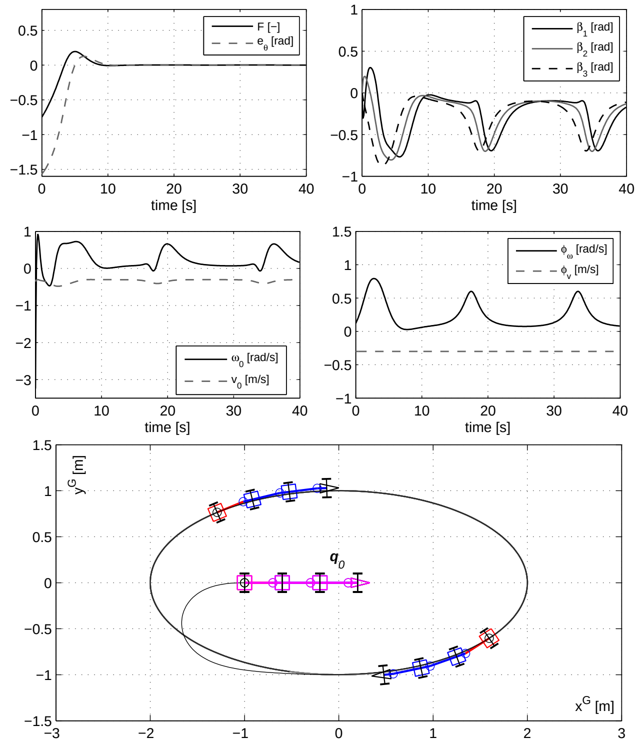

Fig. 3. Sim1: the results for elliptical reference path in the backward motion strategy for nS3T robot with positive hitching offsets ($L_{hi} > 0$, $i = 1, 2, 3$); initial robot configuration $\boldsymbol{q}_0$ has been highlighted in magenta

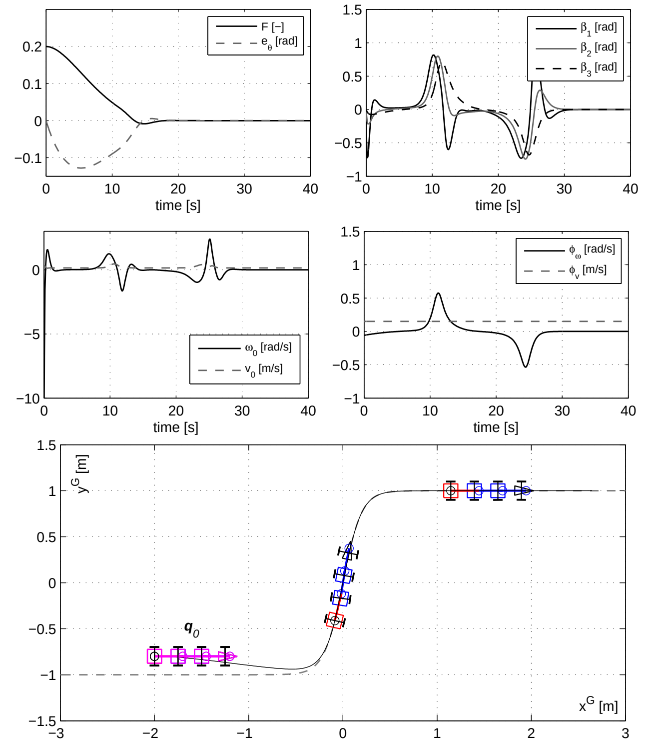

Fig. 4. Sim2: the results for S-like reference path in the forward motion strategy for nS3T robot with negative hitching offsets ($L_{hi} < 0$, $i = 1, 2, 3$); initial robot configuration $\boldsymbol{q}_0$ has been highlighted in magenta

b) Simulations for GNT and SNT robots: Additional two examples, Sim3 and Sim4, illustrate control performance for the approximated cases, where assumption A1 is violated by constructions of the robots. Simulation Sim3 has been conducted with the GNT kinematics where the first and third joints are of on-axle type: $L_{h2} = 0.1\,\text{m}$, $L_{hi} = 0.0$ ($i = 1, 3$). The following parameters and initial conditions have been selected: $\varepsilon_i = 0.01\,\text{m}$ ($i = 1, 3$), $v_d = -0.3\,\text{m/s}$ (backward motion strategy prescribed), $\sigma = -1$, and $\boldsymbol{q}_0 = [0\ \ -1\ \ 0]^\top$. The results of following the elliptical path (23) are shown in Fig. 5. Simulation Sim4 has been performed with the SNT kinematics where all the joints are of on-axle type: $L_{hi} = 0.0\,\text{m}$ ($i = 1, 2, 3$). In this case the parameters and initial values have been taken as follows: $\varepsilon_i = 0.01\,\text{m}$ ($i = 1, 2, 3$), $v_d = -0.15\,\text{m/s}$ (backward motion strategy prescribed), $\sigma = -1$, and $\boldsymbol{q}_0 = [\frac{\pi}{2}\ \ -1\ \ -0.8]^\top$. The results of following the S-like reference path are provided in Fig. 6.

B. Comments to the results

The results obtained for the nominal case in examples Sim1 and Sim2 have shown asymptotic convergence of the path-following error toward zero. One can see continuity and boundedness of the tractor control inputs, and the boundedness of all the joint angles as well. Worth to emphasize that the motion strategy (backward/forward) has been easily selected only by the sign of a value of longitudinal velocity v_d. However, in order to avoid the so-called *vehicle folding effect*, the sign of v_d must be compatible with the sign of the hitching offsets: $\text{sgn}(v_d) = -\text{sgn}(L_{hi})$ (the interested reader is referred to [7] for further details on the vehicle folding phenomenon).

Examples Sim3 and Sim4 illustrate that the control law which utilizes approximations (15) generally leads only to *practical stability* of a closed-loop system, making the error (7) convergent only to some non-zero vicinity of the stable equilibrium. It has been observed that a size of the vicinity is smaller for smaller values of approximation parameters ε_i. Note that through approximation (15), time-evolution of the joint angles is still acceptable, and the control signals remain bounded. Although, one observes relatively high picks in initial angular velocities ($\omega_0(0) \approx 324\,\text{rad/s}$ and $\omega_0(0) \approx -6512\,\text{rad/s}$ for Sim3 and Sim4, respectively). This effect stays in agreement with the upper bound estimated in (13), where for the on-axle joint the Frobenius norm M_j depends on the inverse of approximating parameter ε_j. Further, in example Sim4 where all the transformation matrices had to be approximated by (15), one can observe substantial grow in the noise-sensitivity (here: numerical-sensitivity), which can be seen in the plot of the first joint angle and the angular velocity of the tractor. This sensitivity is a direct consequence of relatively small values selected for parameters ε_i.

For comparison purposes, the error components $|F(x_N, y_N)|$ and $|e_\theta|$ obtained for the four simulation examples, have been presented in Figure 7 in a logarithmic scale. High (nearly exponential) rate of convergence toward zero can be observed only for the nominal cases where the kinematics of nS3T vehicles were used satisfying

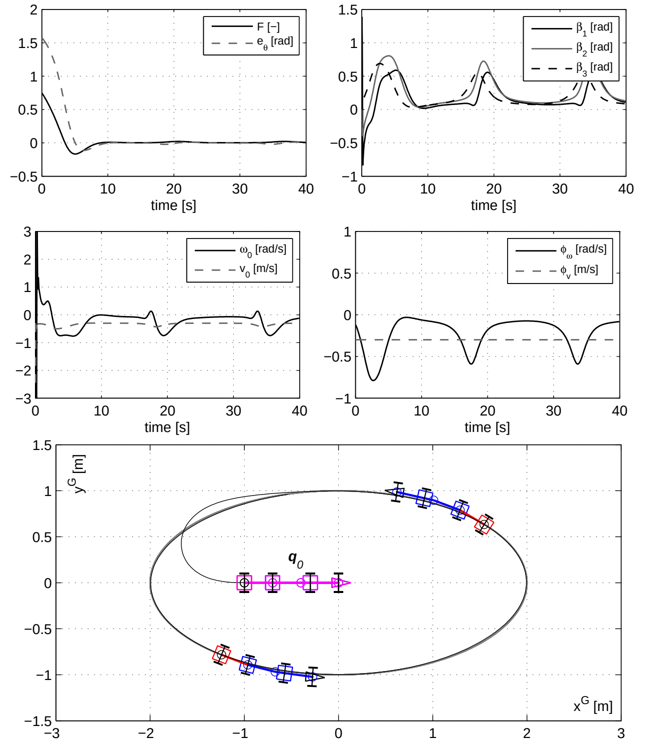

Fig. 5. Sim3: the results for elliptical reference path in the backward motion strategy for G3T robot with the positive second hitching offset ($L_{h2} > 0$), and with zero hitching offsets for the first and third joints ($L_{hi} = 0$, $i = 1, 3$); initial robot configuration $\boldsymbol{q}_0$ has been highlighted in magenta

assumptions A1 and A2.

V. Conclusions

The cascaded control approach to the path-following task has been proposed and numerically tested for N-trailer robots. Decomposition of the control law into the outer-loop unicycle controller and the inner-loop velocity transformation allowed one to use the novel path-following control concept presented in [12]. The latter substantially facilitates execution of the path-following task in comparison to the classical method introduced in [14]. The main benefits of the cascaded control law results from its simplicity and high scalability – a number of trailers influences only a number of matrix multiplications involved in the inner-loop. It has been shown that by simple approximation of the inverse transformation matrices the cascaded control method, originally derived for nSNT robots, can also be applied into GNT and SNT vehicles under the sign-homogeneity assumption for the hitching offsets.

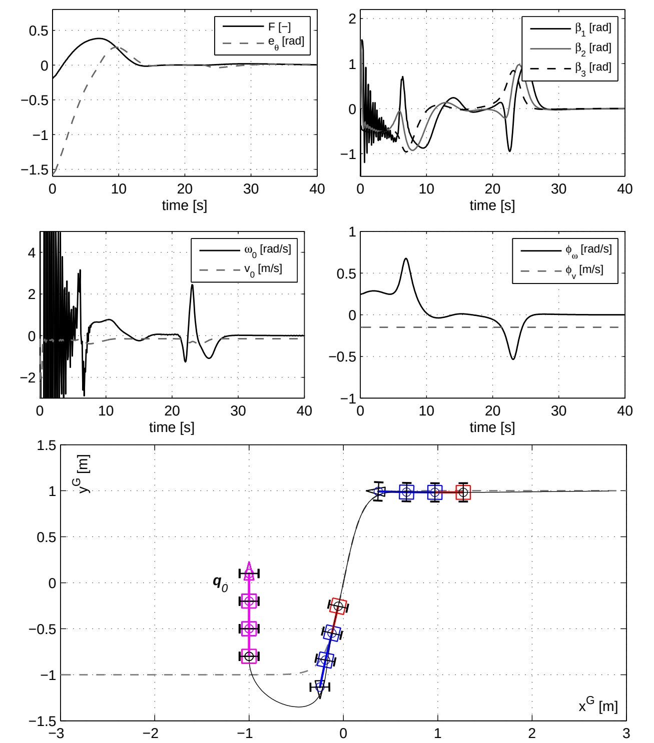

Fig. 6. Sim4: the results for S-like reference path in the backward motion strategy for S3T robot with zero hitching offsets ($L_{hi} = 0$, $i = 1, 2, 3$); initial robot configuration $\boldsymbol{q}_0$ has been highlighted in magenta

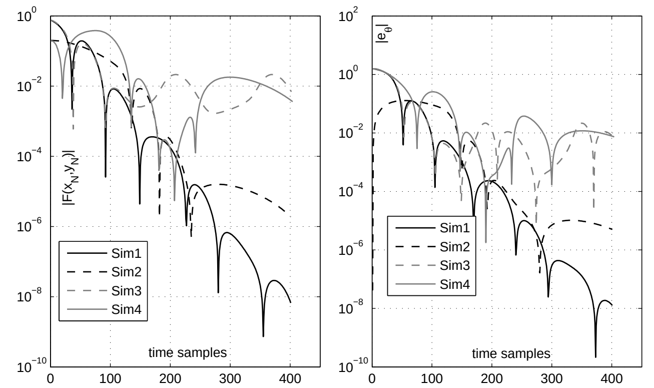

Fig. 7. Comparison of instantaneous distance values $|F(x_N, y_N)|$ (left) and orientation errors $|e_\theta|$ (right) for simulation examples Sim1 to Sim4

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