Finite-Time VFO Stabilizers
For the Unicycle with Constrained Control Input

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Abstract

This chapter presents a formal derivation of a family of the finite-time continuous feedback stabilizers dedicated for a unicycle kinematics in the presence of assumed constraints imposed on the control inputs. Description of the proposed control approach together with the convergence analysis for the closed-loop system are the main part of the work. The result has been illustrated by numerical tests.

1 Introduction

In the recent years, the problem of finite-time convergence for the continuous dynamic systems has attracted more attention of the researchers [1],[4],[2]. On one hand, accomplishing the stabilization control task in a finite time seems to be more natural from a practical point of view than the infinite-time solution characteristic for the *classical* asymptotic results. On the other hand, as presented in [1], the finite-time stable systems are more robust guaranteeing improved rejection of the non-vanishing (persistent) low-level disturbances. Moreover, the continuous finite-time stabilizers allows avoiding the so-called chattering phenomenon intrinsic for example for low-order sliding mode controllers. Utilization of the finite-time stability results to the control design for mobile robots can have an additional merit coming from the possibility of assessing the settling time interval in the closed-loop system. It can facilitate the motion planning stage.
This chapter presents the derivation of a family of the bounded finite-time stabilizers dedicated for a unicycle kinematic model subjected to the constrained control input. The control strategy presented here results from the Vector Field(s) Orientation (VFO) approach described in detail in [3]. The original asymptotic VFO solution presented in [3] has been modified and extended here leading to the finite-time convergent feedback control system preserving the assumed limitations imposed on the control inputs. The formal analysis indicates that the original asymptotic VFO stabilizer can be obtained as a special case of the proposed finite-time controller as will be indicated in the sequel.

2 Prerequisites

Consider a driftless kinematic model of the unicycle, which can be formulated as follows:

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{U} \quad \Rightarrow \quad \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} U_2 \tag{1}$$

where $\boldsymbol{q} = [\theta \ x \ y]^T \in \mathbb{R}^3$ is the unicycle state vector consisting of the orientation angle and the position coordinates expressed in the global frame, respectively. $\boldsymbol{U} =$

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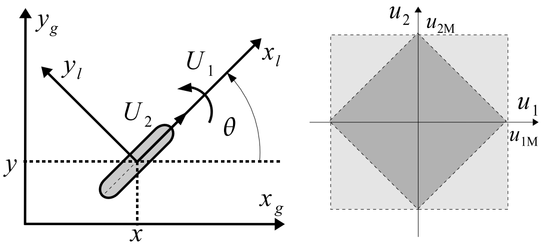

Figure 1: Unicycle in the global frame $\{x_g, y_g\}$ (left) and the set $\mathcal{U}$ of admissible control inputs (right)

$[U_1 \ U_2]^T$ is a general control input vector (compare Fig. 1). For clarity of the subsequent analysis let us introduce the notion of unconstrained, or *nominal*, case of the input taking $\boldsymbol{U} := \boldsymbol{u}_N = [u_{1N} \ u_{2N}] \in \mathbb{R}^2$. The constrained case will be indicated substituting $\boldsymbol{U} := \boldsymbol{u} = [u_1 \ u_2] \in \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^2$ is the bounded set of the admissible control inputs. For the unicycle the set $\mathcal{U}$ has a rectangular shape limited by edges of length $2u_{1M}$ and $2u_{2M}$, where $0 < u_{iM} < \infty, i = 1, 2$ are the maximal admissible values for angular and longitudinal velocity inputs, respectively (see Fig. 1). Note that for the most popular physical realization of the unicycle kinematics in a form of the differentially-driven vehicle, the shape of $\mathcal{U}$ set is different (see the darker area in Fig. 1) and results from the maximal feasible value imposed on a vehicle wheel velocity.

Let us define the control task which will be taken into account in the sequel. For a given reference set-point $\boldsymbol{q}_t = [\theta_t \ x_t \ y_t]^T \in \mathbb{R}^3$ our objective is to design a family of feedback controllers $\boldsymbol{u}(\tau) = \boldsymbol{u}(\boldsymbol{q}_t, \boldsymbol{q}(\tau), \cdot) \in \mathcal{U}$ satisfying the input limitations: $\forall_{\tau \geqslant 0} \ |u_1(\tau)| \leqslant u_{1M}, |u_2(\tau)| \leqslant u_{2M}$ and making the posture error

$$\boldsymbol{e}(\tau) = \begin{bmatrix} e_1(\tau) \\ \boldsymbol{e}^*(\tau) \end{bmatrix} \triangleq \boldsymbol{q}_t - \boldsymbol{q}(\tau), \qquad \boldsymbol{e} \in \mathbb{R}^3 \tag{2}$$

converge in finite time $\tau_f < \infty$ to the assumed vicinity ε of zero in the sense that:

$$\forall_{\tau \geqslant \tau_f} \ e_1(\tau) = 0, \ \|\boldsymbol{e}^*(\tau)\| \leqslant \varepsilon \quad \text{with} \quad \varepsilon \geqslant 0. \tag{3}$$

The mentioned family of controllers one can call as finite-time stabilizers. They will be designed according to the VFO control strategy described in [3] utilizing the finite-time stability result presented in [1].

3 Principles of the VFO control strategy

The VFO control approach originates from the geometrical interpretations connected with a structure of the kinematics (1) (for details see [3]). The description of the VFO concept involves a decomposition of the model (1) as follows:

$$\dot{\theta} = U_1, \tag{4}$$

$$\dot{\boldsymbol{q}}^* = \boldsymbol{g}_2^*(\theta) U_2, \qquad \boldsymbol{g}_2^*(\theta) = [\cos\theta \ \sin\theta]^T, \tag{5}$$

where $[\dot{\theta} \ \dot{\boldsymbol{q}}^{*T}] = \dot{\boldsymbol{q}}$ and $\boldsymbol{q}^* = [x \ y]^T \in \mathbb{R}^2$. Let us also introduce a vector field

$$\boldsymbol{h} = [h_1 \ h_2 \ h_3]^T = \begin{bmatrix} h_1 \\ \boldsymbol{h}^* \end{bmatrix} \in \mathbb{R}^3, \qquad \boldsymbol{h}^* \in \mathbb{R}^2, \tag{6}$$

defined in the tangent space of the unicycle (1). Assume that $\boldsymbol{h}$ is defined in a way that for any state point $\boldsymbol{q}$ the $\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{q}_t, \cdot)$ determines a convergence direction, an orientation and some kind of a distance to the reference point $\boldsymbol{q}_t$. We call $\boldsymbol{h}$

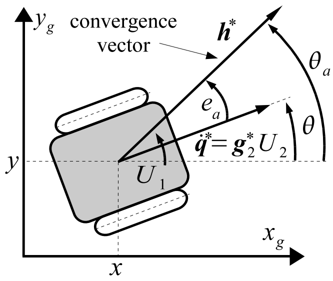

Figure 2: General principles of the VFO control strategy

the convergence vector field, and $\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{q}_t, \cdot)$ the convergence vector. Since $\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{q}_t, \cdot)$ determines the desired time-evolution of the controlled system (from the control task point of view), a natural condition guaranteeing convergence of $\boldsymbol{q}(\tau)$ toward $\boldsymbol{q}_t$ can be written as:

$$[\dot{\boldsymbol{q}}(\tau) - \boldsymbol{h}(\boldsymbol{q}(\tau), \boldsymbol{q}_t, \cdot)] \to \boldsymbol{0} \quad \overset{(4,5,6)}{\Longrightarrow} \quad \left\{ \begin{array}{lcl} u_1(\tau) - h_1(\tau) & \to & 0, \\ \dot{\boldsymbol{q}}^*(\tau) - \boldsymbol{h}^*(\tau) & \to & \boldsymbol{0}. \end{array} \right. \tag{7}$$

The first right-hand-side relation in (7) can be met instantaneously taking

$$U_1(\tau) \stackrel{\Delta}{=} h_1(\tau). \tag{8}$$

The second one in turn can be reduced, after substituting particular components from (5) and (6), to the scalar condition imposed on the θ state variable:

$$[\theta(\tau) - \mathrm{Atan2}\,(\mathrm{sgn}(u_2)h_3(\tau), \mathrm{sgn}(u_2)h_2(\tau))] \to 0, \tag{9}$$

where $\mathrm{sgn}(z) \in \{-1, +1\}$, and $\mathrm{Atan2}\,(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \mapsto (-\pi, \pi]$ is a four-quadrant inverse tangent function. One cannot satisfy (9) instantaneously due to the integral relation represented by (4). Hence, let us introduce an auxiliary variable

$$\theta_a \stackrel{\Delta}{=} \mathrm{Atan2c}\,(\mathrm{sgnU}_2\, h_3, \mathrm{sgnU}_2\, h_2) \in \mathbb{R}, \tag{10}$$

where $\mathrm{Atan2c}\,(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous version of $\mathrm{Atan2}\,(\cdot,\cdot)$ function, and $\mathrm{sgnU}_2 \in \{-1, +1\}$ is a decision variable, which allows freely imposing one of the motion strategies for the unicycle: forward one ($\mathrm{sgnU}_2 = +1$) or backward one ($\mathrm{sgnU}_2 = -1$). Introducing now an auxiliary orientation error

$$e_a(\tau) \stackrel{\Delta}{=} \theta_a(\tau) - \theta(\tau), \qquad e_a \in \mathbb{R}, \tag{11}$$

the problem of satisfying (9) turns into a problem of making (11) converge to zero. Naturally, it can be accomplished by the properly defined U_1 input. As a consequence of (8), the first component of $\boldsymbol{h}$ vector field should define the convergence *direction* for the θ state variable.

It is worth to note that (10) can be alternatively determined as $\theta_a = \arg(\boldsymbol{h}^*) \pm (\pi/2) \cdot (1 - \mathrm{sgnU}_2)$. Using now the form of subsystem (5), where a current direction of time-evolution of $\boldsymbol{q}^*(\tau) = [x(\tau)\ y(\tau)]^T$ depends on a direction of $\boldsymbol{g}_2^*(\theta(\tau))$, one can conclude that in geometrical interpretation making $e_a(\tau)$ equal to zero guarantees putting the direction of $\boldsymbol{g}_2^*(\theta(\tau))$ (and consequently of $\dot{\boldsymbol{q}}^*(\tau)$) on the instantaneous direction determined by the convergence vector $\boldsymbol{h}^*(\boldsymbol{q}, \boldsymbol{q}_t, \cdot)$ (compare Fig. 2, which graphically illustrates general principles of the VFO control strategy). Thus according to (5), to meet the second right-hand-side relation written in (7) it suffices now to design U_2 input in a way that $\mathrm{sgn}(U_2)$ tends to the decision variable sgnU_2 and its absolute value converges to the norm of $\boldsymbol{h}^*$. One can propose the following general definition:

$$U_2(\tau) \stackrel{\Delta}{=} \rho \, \| \boldsymbol{h}^*(\tau) \| \cos\alpha(\tau), \qquad \alpha(\tau) = \angle(\boldsymbol{g}_2^*(\theta(\tau)), \boldsymbol{h}^*(\tau)), \tag{12}$$

where $\rho = \rho(\cdot)$ is a scalar non-negative function which gives an additional degree of freedom for the designer allowing him to shape the magnitude of $U_2(\tau)$ and the convergence rate for the sub-state $\boldsymbol{q}^*(\tau)$ – it will be explained in the sequel.
The general propositions (8) and (12) will get their explicit forms after defining the particular components of the convergence vector field $\boldsymbol{h}$ and the scalar function $\rho(\cdot)$. Since determination of $\boldsymbol{h}$ and $\rho(\cdot)$ is not a unique issue, one can obtain different versions of VFO control laws, or even the whole family of VFO controllers.
Note that $\boldsymbol{h}$ cannot be chosen arbitrarily. It has to be defined properly – let us explain it in a few words. According to the control strategy presented above, the θ state variable plays an auxiliary role in the whole control process – it allows orienting $\boldsymbol{g}_2^*(\theta)$ to put its direction onto the one defined by $\boldsymbol{h}^*(\boldsymbol{q}, \boldsymbol{q}_t, \cdot)$. It is crucial for efficient convergence of $\boldsymbol{q}^*(\tau)$ toward $\boldsymbol{q}_t^*$, but generally does not guarantee that $\theta(\tau)$ will converge to its reference θ_t. The latter must be guaranteed by the proper definition of $\boldsymbol{h}$. It means that $\theta_a = \arg(\boldsymbol{h}^*) \pm (\pi/2) \cdot (1 - \mathrm{sgnU_2})$ has to converge to θ_t in the neighborhood of the reference position $\boldsymbol{q}_t^* = [x_t \ y_t]^T$ (at least in the domain of $\mathbb{S}^1$). As a consequence, the whole state $\boldsymbol{q}$ of the unicycle can be made, in the VFO strategy, convergent to the reference state $\boldsymbol{q}_t$.

In the next sub-chapter the definitions for $\boldsymbol{h}$ and $\rho(\cdot)$, which allows obtaining a family of the finite-time stabilizers for the unicycle kinematics, are proposed.

4 Finite-time VFO stabilizers

Let us first introduce the definitions of the convergence vector field and the VFO control inputs u_{1N} and u_{2N} for the *nominal* (unconstrained) case. One can propose to take:

$$h_1(\tau) \triangleq k_1 \mathrm{sign}(e_a(\tau))\, |e_a(\tau)|^\delta + \dot{\theta}_{aN}(\tau), \qquad \delta \in (0,1), \tag{13}$$

$$\boldsymbol{h}^*(\tau) \triangleq k_p \boldsymbol{e}^*(\tau) + \boldsymbol{v}^*(\tau), \tag{14}$$

where $\mathrm{sign}(0) = 0$, and

$$\boldsymbol{v}^*(\tau) = [v_x(\tau) \ v_y(\tau)]^T \triangleq -\eta \mathrm{sgnU_2} \,\|\, \boldsymbol{e}^*(\tau)\| \,\boldsymbol{g}_{2t}^*, \qquad \boldsymbol{g}_{2t}^* = [\cos\theta_t \ \sin\theta_t]^T, \tag{15}$$

$$\dot{\theta}_{aN}(\tau) = \frac{1}{\|\, \boldsymbol{h}^*(\tau)\|^2} \left[\dot{h}_{3N}(\tau) h_2(\tau) - h_3(\tau)\dot{h}_{2N}(\tau)\right] \quad \text{for} \quad \|\, \boldsymbol{h}^*(\tau)\| \neq 0 \tag{16}$$

with $k_1, k_p > 0$ and $0 < \eta < k_p$ being the VFO design coefficients. The feed-forward term $\dot{\theta}_{aN}$ determined in (16) results from the time-differentiation of the auxiliary variable (10) and is denoted by the subscript N, since the time-derivatives $\dot{h}_{2N}$ and $\dot{h}_{3N}$ from the right-hand side are computed using the *nominal* control input u_{2N} (see (37) and (38) in Appendix). Now, according to (8) and using (12) we obtain the *nominal* VFO control inputs in the following form:

$$u_{1N}(\tau) := k_1 \mathrm{sign}(e_a(\tau))\, |e_a(\tau)|^\delta + \dot{\theta}_{aN}(\tau), \qquad \delta \in (0,1), \tag{17}$$

$$u_{2N}(\tau) := \rho \,\|\, \boldsymbol{h}^*(\tau)\| \cos\alpha(\tau), \tag{18}$$

where the function $\rho = \rho(\cdot)$ will be determined further. Note that the $\mathrm{sign}(e_a)$ term included in (17) does not lead to the chattering phenomenon characteristic for the sliding mode controllers, since the exponent $\delta = 0$ has been excluded[1] from the set proposed in (17). On the other hand, the terms (10) and (16), and consequently input (17), are not determined for $\|\, \boldsymbol{h}^*\| = 0$. Hence, proposition (17) involves additional definitions for θ_a and $\dot{\theta}_{aN}$ in the assumed neighborhood of $\boldsymbol{h}^* = \boldsymbol{0}$ – it will be taken into account in the final definition of the VFO control law.

[1] The case $\delta = 0$ leads to the classical first-order sliding mode control. The other special case, excluded here as well, is for $\delta = 1$ yielding the linear proportional controller – this case was used in the original VFO formulation in [3] leading to the asymptotic convergence for $e_a(\tau)$.

In general, one cannot guarantee that control inputs (17) and (18) will stay in the admissible set $\mathcal{U}$ of the control space. Because we are here interested in the constrained case of the control law, one has to describe an input scaling procedure, which allows fulfilling the control input limitations u_{1M}, u_{2M} connected with set $\mathcal{U}$. Let us formulate the scaling procedure for the rectangular set $\mathcal{U}$ denoted in Fig. 1. The output of the procedure is a scaled control vector

$$\boldsymbol{u}(\tau) = s(\tau) \cdot \boldsymbol{u}_N(\tau), \qquad \boldsymbol{u} \in \mathcal{U} \tag{19}$$

where $s(\tau)$ is a scaling function

$$s(\tau) \triangleq \frac{1}{d(\tau)}, \qquad d(\tau) = \max\left\{\frac{|u_{1N}(\tau)|}{u_{1M}}; \frac{|u_{2N}(\tau)|}{u_{2M}}; 1\right\} \quad \Rightarrow \quad s(\tau) \in (0, 1] \tag{20}$$

computed using the *nominal* control inputs taken from (17)-(18). The output of the above procedure is the constrained control input $\boldsymbol{u}(\tau) = [s(\tau)\, u_{1N}(\tau)\ s(\tau)\, u_{2N}(\tau)]^T$ belonging to the set $\mathcal{U}$ for all $\tau \geqslant 0$. Note that after scaling, the direction of the *nominal* control input is preserved ($\boldsymbol{u} \| \boldsymbol{u}_N$). From now on, the aim is to analyze, if the VFO control law with the constrained inputs (19), applied to the unicycle model (1) with the properly defined function $\rho(\cdot)$ guarantees stability of the closed-loop system and the finite-time convergence of the posture error (2) to the assumed neighborhood of zero. To make our statements strict enough, let us formulate the following proposition.

Proposition 1 *The VFO control law*

$$u_1(\tau) := s(\tau) \cdot \left[k_1 \mathrm{sign}(e_a(\tau))\, |e_a(\tau)|^{\delta} + \dot{\theta}_{aN}(\tau)\right], \qquad \delta \in (0, 1), \tag{21}$$

$$u_2(\tau) := s(\tau) \cdot \rho \, \| \boldsymbol{h}^*(\tau)\| \cos \alpha(\tau), \tag{22}$$

with $\mathrm{sign}(0) = 0$, $e_a(\tau)$, $\boldsymbol{h}^*(\tau)$ *and* $s(\tau)$ *determined in (11), (14) and (20), respectively, where*

$$\theta_a(\tau), \dot{\theta}_{aN}(\tau) := \begin{cases} (10),\ (16) & \text{for} \quad \| \boldsymbol{e}^*(\tau)\| > \varepsilon \\ \theta_t,\ \ 0 & \text{for} \quad \| \boldsymbol{e}^*(\tau)\| \leqslant \varepsilon \end{cases} \tag{23}$$

$$\rho := \begin{cases} \frac{\rho_0}{\| \boldsymbol{h}^*(\tau)\|} \, \| \boldsymbol{e}^*(\tau)\|^{\beta} & \text{for} \quad \| \boldsymbol{e}^*(\tau)\| > \varepsilon \\ 0 & \text{for} \quad \| \boldsymbol{e}^*(\tau)\| \leqslant \varepsilon \end{cases} \tag{24}$$

with $\rho_0 > 0$ *and* $\beta \in [0, 1)$*, applied to the unicycle kinematics (1) implies that for any given reference point* $\boldsymbol{q}_t \in \mathbb{R}^3$ *and any bounded initial condition* $\boldsymbol{e}(0) \in \mathbb{R}^3$ *the posture error* $\boldsymbol{e}(\tau)$ *converges in the finite time* τ_f *to the arbitrarily small vicinity* $\varepsilon \geqslant 0$ *in the sense of (3), guaranteeing that the control input* $\boldsymbol{u} = [u_1\ u_2]^T$ *stays in the assumed admissible closed set* $\mathcal{U}$ *determined by the bounds* u_{1M}, u_{2M}.

One can note that the above proposition describes in fact a family of VFO controllers obtained for different values of δ and β exponents.

Let us now turn to the convergence analysis. According to conditions (23)-(24) which are determined for two regions of the position error $\boldsymbol{e}^*$, consider first the behavior of the closed-loop system for the case when $\| \boldsymbol{e}^*(\tau)\| > \varepsilon$ showing that $\| \boldsymbol{e}^*(\tau)\|$ converges in a finite time to the assumed vicinity ε. Next, evolution of the closed-loop system when $\| \boldsymbol{e}^*(\tau)\| \leqslant \varepsilon$ will be analyzed.

Analysis under condition: $\| \boldsymbol{e}^*(\tau)\| > \varepsilon$. Let us take into account behavior of the auxiliary orientation error $e_a(\tau)$ by introducing the positive-definite function $V_a = \frac{1}{2} e_a^2$. Its time-derivative can be calculated as follows:

$$\dot{V}_a = e_a \dot{e}_a = e_a(\dot{\theta}_a - \dot{\theta}) \stackrel{(1)}{=} e_a(s\dot{\theta}_{aN} - u_1),$$

where we have used the fact that $s\dot{\theta}_{aN} \equiv \dot{\theta}_a$ (see (39) in Appendix) and for the constrained case we have $U_1 := u_1$ and $U_2 := u_2$, respectively. Using proposition

(21) one gets:

$$\dot{V}_a = e_a(s\dot{\theta}_{aN} - sk_1\mathrm{sign}(e_a)\,|e_a|^{\delta} - s\dot{\theta}_{aN}) = -s\,k_1\,|e_a|^{\delta+1} \leqslant -\underline{s}\,k_1\sqrt{2}^{\delta+1}\cdot V_a^{(\delta+1)/2},$$

where

$$\underline{s} \stackrel{\Delta}{=} \inf_{\tau}[s(\tau)] \quad \stackrel{(20)}{\Longrightarrow} \quad \underline{s} \in (0,1]. \tag{25}$$

The result presented in [1] allows concluding the finite-time convergence of $e_a(\tau)$ to zero: $\lim_{\tau\to\tau_a} e_a(\tau) = 0$ within the time

$$\tau_a \leqslant \frac{2}{c_a(1-\delta)}\overline{V}_a^{(1-\delta)/2}, \qquad c_a = \underline{s}\,k_1\sqrt{2}^{\delta+1}, \quad \overline{V}_a = \frac{1}{2}e_a^2(0). \tag{26}$$

This partial result will be utilized in the subsequent analysis. Let us now consider the time-evolution of the position error $\boldsymbol{e}^*(\tau)$. For the constant reference point we have $\dot{\boldsymbol{e}}^* = -\dot{\boldsymbol{q}}^* = -\boldsymbol{g}_2^*(\theta)U_2$, which for the constrained case ($U_2 := u_2$) yields $\dot{\boldsymbol{e}}^* = -\boldsymbol{g}_2^*(\theta)u_2 = -s\,\boldsymbol{g}_2^*(\theta)u_{2N}$. The latter equation can be rewritten alternatively in the following form: $\dot{\boldsymbol{e}}^* = -s\,\boldsymbol{g}_2^*(\theta)u_{2N} + s\rho\boldsymbol{h}^* - s\rho(k_p\boldsymbol{e}^* + \boldsymbol{v}^*)$ leading to the following differential equation:

$$\dot{\boldsymbol{e}}^* + s\rho k_p\boldsymbol{e}^* = s\rho\boldsymbol{r} - s\rho\boldsymbol{v}^*, \tag{27}$$

where

$$\boldsymbol{r} = \boldsymbol{h}^* - \boldsymbol{g}_2^*(\theta)\overline{u}_{2N}, \qquad \text{and} \qquad \overline{u}_{2N} = \frac{u_{2N}}{\rho}. \tag{28}$$

We can show (see Appendix) that the following two relations hold:

$$\|\,\boldsymbol{r}\| = \|\,\boldsymbol{h}^*\|\,\gamma(\theta), \qquad\qquad \lim_{\theta\to\theta_a}\gamma(\theta) = 0, \tag{29}$$

where $\gamma(\theta) = \sqrt{1-\cos^2\alpha(\theta)} \in [0,1]$. Introducing the positive-definite function $V = \frac{1}{2}\boldsymbol{e}^{*T}\boldsymbol{e}^*$ one can estimate its time-derivative as follows:

$$\begin{aligned}
\dot{V} &= \boldsymbol{e}^{*T}\dot{\boldsymbol{e}}^* = \boldsymbol{e}^{*T}\left[-s\rho k_p\boldsymbol{e}^* + s\rho\boldsymbol{r} - s\rho\boldsymbol{v}^*\right] = \\
&= -s\rho k_p\,\|\,\boldsymbol{e}^*\|^2 + s\rho\boldsymbol{e}^{*T}\boldsymbol{r} - s\rho\boldsymbol{e}^{*T}\boldsymbol{v}^* \leqslant \\
&\leqslant -s\rho\left[k_p\,\|\,\boldsymbol{e}^*\|^2 - \|\,\boldsymbol{e}^*\|\,\|\,\boldsymbol{r}\| - \|\,\boldsymbol{e}^*\|\,\|\,\boldsymbol{v}^*\|\right] \leqslant \\
&\leqslant -s\rho\left[k_p\,\|\,\boldsymbol{e}^*\|^2 - \|\,\boldsymbol{e}^*\|\,\|\,\boldsymbol{h}^*\|\,\gamma - \eta\,\|\,\boldsymbol{e}^*\|^2\right] = \\
&= -s\rho\left[k_p\,\|\,\boldsymbol{e}^*\|^2 - \|\,\boldsymbol{e}^*\|\,\|\,k_p\boldsymbol{e}^* + \boldsymbol{v}^*\|\,\gamma - \eta\,\|\,\boldsymbol{e}^*\|^2\right] \leqslant \\
&\leqslant -s\rho\left[k_p\,\|\,\boldsymbol{e}^*\|^2 - \gamma k_p\,\|\,\boldsymbol{e}^*\|^2 - \gamma\eta\,\|\,\boldsymbol{e}^*\|^2 - \eta\,\|\,\boldsymbol{e}^*\|^2\right] = \\
&= -s\rho\left[(k_p-\eta) - \gamma(k_p+\eta)\right]\|\,\boldsymbol{e}^*\|^2 = -s\rho\zeta(\gamma)\,\|\,\boldsymbol{e}^*\|^2.
\end{aligned}$$

One can now use the definition (24) of function ρ rewriting it as

$$\rho = \frac{\rho_0}{\|\,\boldsymbol{\vartheta}\|}\,\|\,\boldsymbol{e}^*\|^{\beta-1}, \tag{30}$$

where

$$\boldsymbol{\vartheta}(\tau) = k_p\boldsymbol{\vartheta}_e(\tau) - \eta\mathrm{sgnU}_2\boldsymbol{g}_{2t}^* \qquad \text{and} \qquad \forall_{\tau\geqslant 0}\ \|\,\boldsymbol{\vartheta}(\tau)\| \neq 0, \tag{31}$$

which in turn follows from equations (14) and (15), since

$$\|\,\boldsymbol{h}^*\| = \|\,k_p\boldsymbol{e}^* - \eta\mathrm{sgnU}_2\,\|\,\boldsymbol{e}^*\|\cdot\boldsymbol{g}_{2t}^*\| = \|\,\boldsymbol{e}^*\|\cdot\|\,\boldsymbol{\vartheta}\| \qquad \text{and} \qquad \boldsymbol{\vartheta}_e = \frac{\boldsymbol{e}^*}{\|\,\boldsymbol{e}^*\|}. \tag{32}$$

Continuing estimating the upper bound of the time-derivative $\dot{V}$ one gets:

$$\begin{aligned}
\dot{V} &\leqslant -s\frac{\rho_0}{\|\,\boldsymbol{\vartheta}\|}\,\|\,\boldsymbol{e}^*\|^{\beta-1}\cdot\zeta(\gamma)\,\|\,\boldsymbol{e}^*\|^2 = -s\frac{\rho_0}{\|\,\boldsymbol{\vartheta}\|}\zeta(\gamma)\,\|\,\boldsymbol{e}^*\|^{\beta+1} \leqslant \\
&\leqslant -\underline{s}\frac{\rho_0}{k_p+\eta}\zeta(\gamma)\,\|\,\boldsymbol{e}^*\|^{\beta+1} = -\sqrt{2}^{\beta+1}\,\frac{\underline{s}\,\rho_0}{k_p+\eta}\zeta(\gamma)\cdot V^{(\beta+1)/2},
\end{aligned}$$

where $\underline{s}$ has been defined in (25). The right-hand side of the above inequality will be negative definite if the function $\zeta(\gamma)$ is positive. It leads to the following convergence condition for the position error $\boldsymbol{e}^*$:

$$\zeta(\gamma) > 0 \quad \Leftrightarrow \quad \gamma(\tau) < \Gamma, \qquad \Gamma = \frac{k_p - \eta}{k_p + \eta}. \tag{33}$$

Since $\eta < k_p$ (from assumption) the ratio $\Gamma < 1$. Even for some finite-time interval $[0, \tau_\gamma)$ when $\zeta(\gamma(\tau)) < 0$, the time-derivative $\dot{V}$ is bounded, so the norms $\|\boldsymbol{e}^*\|$, $\|\dot{\boldsymbol{e}}^*\|$ are bounded as well (finite-time escape is not possible). Moreover, since $\gamma(\theta(\tau)) \in [0, 1]$ for all $\tau \geqslant 0$ and due to the limit from (29) together with the finite-time convergence result of $e_a(\tau)$ to zero allows concluding what follows:

$$\exists_{\tau_\gamma < \tau_a} : \forall_{\tau \geqslant \tau_\gamma}\ \gamma(\tau) < \Gamma \qquad \text{and} \qquad \forall_{\tau \geqslant \tau_a}\ \gamma(\tau) = 0 \quad \Rightarrow \quad \forall_{\tau \geqslant \tau_a}\ \zeta(\gamma) = k_p - \eta.$$

The above reasoning together with the convergence result presented in [1] yields the finite-time convergence for $\|\boldsymbol{e}^*(\tau)\|$ to the assumed vicinity $\varepsilon \geqslant 0$ in finite time τ_e in the sense that:

$$\lim_{\tau \to \tau_e} \|\boldsymbol{e}^*(\tau)\| = \varepsilon, \qquad \text{where} \qquad \tau_e \leqslant \tau_a + \frac{2}{c_e(1-\beta)} \overline{V}^{(1-\beta)/2} \tag{34}$$

with $\overline{V} = 0.5\,\|\boldsymbol{e}^*(\tau_a)\|^2$ and $c_e = \underline{s}\,\rho_0 \frac{k_p - \eta}{k_p + \eta} \sqrt{2}^{\beta+1}$.

Analysis under condition: $\|\boldsymbol{e}^*(\tau)\| \leqslant \varepsilon$. In this case $u_2 = 0$ according to (24), hence $\dot{\boldsymbol{e}}^* \equiv 0$ and consequently $\|\boldsymbol{e}^*(\tau)\| = \varepsilon$ for all $\tau \geqslant \tau_e$. Additionally, due to (23) one obtains for $\tau \geqslant \tau_e$ the following differential equation for the orientation angle dynamics: $\dot{e}_1(\tau) = -k_1 \mathrm{sign}(e_1(\tau))\, |e_1(\tau)|^\delta$ yielding:

$$\lim_{\tau \to \tau_1} e_1(\tau) = 0, \quad \text{where} \quad \tau_1 = \tau_e + \frac{|e_1(\tau_e)|^{1-\delta}}{k_1(1-\delta)}. \tag{35}$$

Finally, using the partial results from (26), (34) and (35) one can conclude that the convergence claim (3) holds for the finite time

$$\tau_f \ \leqslant \ \left[\frac{2}{c_a(1-\delta)} \overline{V}_a^{(1-\delta)/2} + \frac{2}{c_e(1-\beta)} \overline{V}^{(1-\beta)/2} + \frac{|e_1(\tau_e)|^{1-\delta}}{k_1(1-\delta)} \right]. \tag{36}$$

Remark 1 *Substituting (24) (for the case* $\|\boldsymbol{e}^*\| > \varepsilon$*) into (22) yields a simpler form of the* u_2 *input, namely:* $u_2(\tau) = s(\tau) \cdot \rho_0 \|\boldsymbol{e}^*(\tau)\|^\beta \cos\alpha(\tau)$*. For* $\beta = 0$ *it gives a signal shaped by* ρ_0 *and dynamically scaled by* $s(\tau)$*. Note also that for the special case, taking* $\delta = \beta = 0$ *and* $\rho_0 = \|\boldsymbol{h}^*\|$*, the definitions (21)-(22) reduces to the asymptotic VFO stabilizer described in [3].*

Remark 2 *The assumed vicinity* ε *used in (3) and (23)-(24) can be theoretically taken equal to zero leading to the stabilization result with a perfect precision for the whole posture error* $\boldsymbol{e}$*. In this case it can be shown that (21) and (22) are the continuous signals but locally non-Lipschitz near* $e_a = 0$ *and* $\|\boldsymbol{e}^*\| = 0$ *, respectively (unavoidable for the finite-time convergence [4]). In practical implementation however, due to the measurement noises and additional dynamics of the robot platform not taken into account in (1), the sufficiently small but non-zero value for* ε *will be preferred.*

5 Numerical tests

Preliminary simulation tests has been conducted for the parallel parking task with the precision imposed by the vicinity $\varepsilon = 0.001$ m, where the reference point $\boldsymbol{q}_t = \boldsymbol{0}$ and the initial condition $\boldsymbol{q}(0) = [0\ 0\ 2]^T$ for the unicycle have been assumed. Values

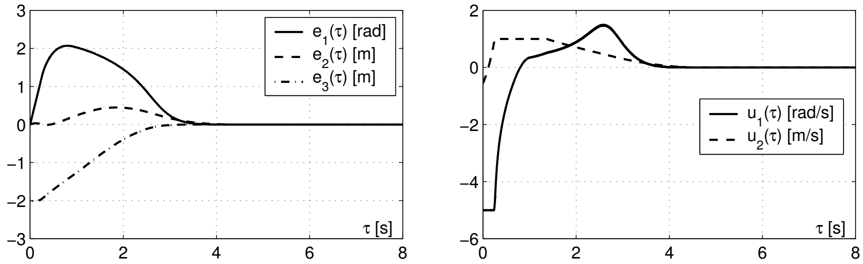

Figure 3: Time plots of the posture errors (left) and the constrained control inputs (right)

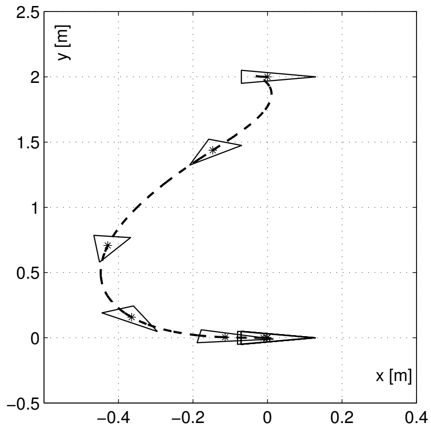

Figure 4: Geometrical path drawn by the unicycle during parallel parking maneuvers

of the particular controller parameters have been chosen as follows: $k_1 = 5$, $k_p = 3$, $\eta = 2$, $\delta = \beta = 2/3$, $\rho_0 = 1$. Moreover, the bounds of the admissible control set have been taken as: $u_{1M} = 5\,\mathrm{rad/s}$ and $u_{2M} = 1$ m/s. The decision variable $\mathrm{sgnU}_2 := +1$ has determined the forward motion strategy for the unicycle. The control quality obtained for the finite-time VFO stabilizer has been illustrated in Figs. 3 and 4. It is worth to note that the control inputs preserve the control bounds within the whole control time-horizon, and the posture errors converge to zero in the finite-time $\tau_f \approx 4.36\,\mathrm{s}$.

6 Conclusions

In the presented chapter the family of finite-time VFO stabilizers has been derived for the unicycle kinematic model in the case where the control inputs have been constrained to the assumed admissible and bounded control set $\mathcal{U} \subsetneq \mathbb{R}^2$. The control law described in this paper is an extension of the original asymptotic VFO stabilizer, which can be retrieved as a special case of the presented solution. It has been shown that the proposed control strategy together with the input scaling procedure lead to the finite-time posture error convergence to the assumed arbitrarily small, including zero, vicinity of the origin guaranteeing that the control inputs stay in the admissible control set $\mathcal{U}$.

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Appendix

Derivation of the left-hand side of (29). Recalling (28) one can write:

$$\boldsymbol{r} = \boldsymbol{h}^* - \boldsymbol{g}_2^* \overline{u}_{2N} = \begin{bmatrix} h_2 \\ h_3 \end{bmatrix} - \begin{bmatrix} \overline{u}_{2N} \cos\theta \\ \overline{u}_{2N} \sin\theta \end{bmatrix} \overset{(18)}{=} \|\boldsymbol{h}^*\| \begin{bmatrix} \frac{h_2}{\|\boldsymbol{h}^*\|} - \cos\alpha\cos\theta \\ \frac{h_3}{\|\boldsymbol{h}^*\|} - \cos\alpha\sin\theta \end{bmatrix}.$$

Now one can obtain what follows (using the notation $\mathrm{c}\beta \equiv \cos\beta$, $\mathrm{s}\beta \equiv \sin\beta$ for compactness):

$$\begin{aligned} \|\boldsymbol{r}\|^2 &= \|\boldsymbol{h}^*\|^2 \left[\frac{h_2^2}{\|\boldsymbol{h}^*\|^2} - \frac{2h_2\mathrm{c}\alpha\mathrm{c}\theta}{\|\boldsymbol{h}^*\|} + \mathrm{c}^2\alpha\mathrm{c}^2\theta + \frac{h_3^2}{\|\boldsymbol{h}^*\|^2} - \frac{2h_3\mathrm{c}\alpha\mathrm{s}\theta}{\|\boldsymbol{h}^*\|} + \mathrm{c}^2\alpha\mathrm{s}^2\theta \right] = \\ &= \|\boldsymbol{h}^*\|^2 \left[1 - 2\mathrm{c}\alpha \frac{h_2\mathrm{c}\theta + h_3\mathrm{s}\theta}{\|\boldsymbol{h}^*\|} + \mathrm{c}^2\alpha \right] = \|\boldsymbol{h}^*\|^2 \left(1 - 2\mathrm{c}\alpha\mathrm{c}\alpha + \mathrm{c}^2\alpha\right) = \|\boldsymbol{h}^*\|^2 \left(1 - \mathrm{c}^2\alpha\right) \end{aligned}$$

and finally $\|\boldsymbol{r}\| = \|\boldsymbol{h}^*\| \sqrt{1 - \cos^2\alpha(\theta)} = \|\boldsymbol{h}^*\| \gamma(\theta)$.

Derivation of the limit from (29). Since $\cos\alpha = (\boldsymbol{g}_2^{*T}(\theta)\boldsymbol{h}^*)/(\left\|\boldsymbol{g}_2^*(\theta)\right\| \|\boldsymbol{h}^*\|)$ one can obtain:

$$\gamma^2(\theta) \quad = \quad 1 - \mathrm{c}^2\alpha(\theta) = 1 - \frac{(h_2\mathrm{c}\theta + h_3\mathrm{s}\theta)^2}{\|\boldsymbol{h}^*\|^2 \left\|\boldsymbol{g}_2^*\right\|^2} = \frac{(h_2\mathrm{s}\theta - h_3\mathrm{c}\theta)^2}{h_2^2 + h_3^2}.$$

At the limit $\theta(\tau) \to \theta_a(\tau)$ we have according to (10): $\lim_{\theta\to\theta_a} \tan\theta = h_3/h_2 \Rightarrow \lim_{\theta\to\theta_a} \mathrm{s}\theta = (h_3\mathrm{c}\theta)/h_2$, which substituted into the above equation gives $\lim_{\theta\to\theta_a} \gamma(\theta) = 0$.

Explanation of the particular velocity terms in (16). Recalling that $\boldsymbol{q}_t = const. \Rightarrow \dot{\boldsymbol{e}}^* = -\dot{\boldsymbol{q}}^*$ and according to (14), (15), (2) and (1) one can obtain:

$$\dot{h}_{2N} = -k_p\dot{x} + \dot{v}_x = u_{2N}\left(-k_p\cos\theta + \eta\,\mathrm{sgnU}_2 \frac{\boldsymbol{e}^{*T}\boldsymbol{g}_2^*}{\|\boldsymbol{e}^*\|}\cos\theta_t \right), \tag{37}$$

$$\dot{h}_{3N} = -k_p\dot{y} + \dot{v}_y = u_{2N}\left(-k_p\sin\theta + \eta\,\mathrm{sgnU}_2 \frac{\boldsymbol{e}^{*T}\boldsymbol{g}_2^*}{\|\boldsymbol{e}^*\|}\sin\theta_t \right), \tag{38}$$

where $\dot{x} = u_{2N}\cos\theta$ and $\dot{y} = u_{2N}\sin\theta$ have been used according to (1) with $U_2 := u_{2N}$. Recalling now (16) together with (37) and (38) rewritten as $\dot{h}_{2N} = u_{2N}H_2$, $\dot{h}_{3N} = u_{2N}H_3$ allows one to write:

$$s \cdot \dot{\theta}_{aN} = \frac{s \cdot u_{2N}H_3h_2 - s \cdot u_{2N}H_2h_3}{h_2^2 + h_3^2} \overset{(19)}{=} \frac{u_2H_3h_2 - u_2H_2h_3}{h_2^2 + h_3^2} = \frac{\dot{h}_3h_2 - \dot{h}_2h_3}{h_2^2 + h_3^2} \equiv \dot{\theta}_a. \tag{39}$$

Hence, the term $\dot{\theta}_a$ describes the time-derivative of θ_a in the case, where the particular time-derivatives $\dot{h}_2$ and $\dot{h}_3$ are computed with the scaled (constrained) input u_2 defined in (22).

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