Complementary note on:

Solution of a first-order differential equation with decaying *gain*

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Abstract

This note aims at complementing the formal analysis provided in papers [1] and [2] in relation to terminal convergence of a first-order differential equation with decaying *gain*. Considerations included in this note leads to the correction of an improper corollary formulated in [2] with respect to equation[1] {31}.

1 General considerations

Let us consider the first-order differential equation of the form

$$\dot{x}(t) = -c\, g(t)\, \sin x(t), \tag{1}$$

where $c > 0$ is a positive constant, t is time variable, and $g(t)$ is a bounded nonnegative function such that $g(t) \to 0$ as $t \to 0$. The term $c\, g(t)$ will be called the *gain*, which in our case terminally tends to zero (decays). Solution of (1) for $x \in [-\pi, \pi]$ can be written as

$$x(t) = 2 \arctan\left[X_0 \cdot \exp(-cI(0,t))\right], \quad t \geq 0, \tag{2}$$

where $X_0 = \tan \frac{x(0)}{2}$, and

$$I(0,t) \triangleq \int_0^t g(\xi) d\xi. \tag{3}$$

We are interested in the terminal behavior of solution (2) as $t \to \infty$. It directly depends on integral (3) evaluated in infinity, namely $I(0, \infty)$. The terminal convergence of $x(t)$ to zero (for $t \to \infty$) requires $I(0, \infty) = \infty$, which could be obtained for instance if $g(t) = g$ =const. But it is not the case here, since we assume $g(t) \to 0$. The fact that $g(t \to \infty) \to 0$ does not necessarily preclude convergence of $x(t)$ to zero. It depends of the rate of convergence of function $g(t)$. Thus, to make an appropriate conclusion about terminal behavior of $x(t)$ one has to investigate integrability of function $g(t)$.

2 Application of the above result to papers [2] and [1]

In paper [2], equation {31} is in the form of (1) by taking $x := e_\theta$, $c := -\text{sgn}(e_{x0})/L_1$, and $g(t) := \|\boldsymbol{h}^*(t)\|$. To check if the error $e_\theta(t)$ terminally converges to zero we must investigate integrability of $\|\boldsymbol{h}^*(t)\|$ (according to the result (2)-(3)). Recalling the results presented in [2] we know that $\boldsymbol{h}^* = \boldsymbol{h}^*(t) = k_p \boldsymbol{e}^*(t) - \eta\sigma \|\boldsymbol{e}^*(t)\| \boldsymbol{g}_2^*(\beta_t)$, thus (note: $\|\boldsymbol{g}_2^*(\beta_t)\| \equiv 1$ and $\sigma \in \{-1, +1\}$)

$$\|\boldsymbol{h}^*(t)\| \leq (k_p + \eta) \|\boldsymbol{e}^*(t)\| = b\, \|\boldsymbol{e}^*(t)\|, \qquad \eta \in (0, k_p). \tag{4}$$

Since $b = (k_p + \eta) > 0$ is a constant, integrability of $\|\boldsymbol{h}^*(t)\|$ is equivalent to integrability of $\|\boldsymbol{e}^*(t)\|$. According to the results presented in [3] (cf. page 53), for the set-point control task with the VFO controller one can write

$$\begin{aligned} I(0,\infty) &= \int_0^\infty \|\boldsymbol{h}^*(t)\|\, dt \\ &\leq b \int_0^\infty \|\boldsymbol{e}^*(t)\|\, dt = b \int_0^{\tau_\gamma} \|\boldsymbol{e}^*(t)\|\, dt + b \int_{\tau_\gamma}^\infty \|\boldsymbol{e}^*(t)\|\, dt \leq E_\gamma + b \int_{\tau_\gamma}^\infty \|\boldsymbol{e}^*(\tau_\gamma)\| \exp(-\zeta_\gamma (t - \tau_\gamma)) dt, \end{aligned} \tag{5}$$

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[1] References to particular equations taken from papers [1–3] are indicated by numbers in {·} brackets.

where $0 \leq E_\gamma < \infty$, $\tau_\gamma \geq 0$, $\zeta_\gamma > 0$, and $\| \boldsymbol{e}^*(\tau_\gamma)\| < \infty$. As a consequence, integral (5) is bounded, since

$$I(0,\infty) \leq E_\gamma + b\,\| \boldsymbol{e}^*(\tau_\gamma)\| / \zeta_\gamma. \tag{6}$$

Hence, one concludes that $I(0,\infty) < \infty$, and error $e_\theta(t)$ terminally **does not converge to zero** – in contrast to the incorrect statement about *local asymptotic stability of equilibrium* $e_{\theta 1E} = 0$ formulated in [2] on page 270 under equation {31}. Terminal value of e_θ depends on the value of integral (3) at infinity (for $t \to \infty$).

Similar arguments apply to the analysis done in paper [1] on page 512 where the integral defined by {64} has been considered (cf. also Remark 2 on page 512). Upon {64} one can write[2]

$$\begin{aligned} I(\tau_d,\tau) &= \int_{\tau_d}^{\tau} s(\xi)\,\| \boldsymbol{h}^*(\bar{\boldsymbol{e}}(\xi))\| \cos e_a(\xi) d\xi \leq \int_{\tau_d}^{\tau} \| \boldsymbol{h}^*(\bar{\boldsymbol{e}}(\xi))\|\, d\xi \\ &\leq \int_{\tau_d}^{\tau} (k_p+\eta)\,\| \bar{\boldsymbol{e}}^*(\xi)\|\, d\xi = (k_p+\eta)\int_{\tau_d}^{\tau} \| \bar{\boldsymbol{e}}^*(\xi)\|\, d\xi, \end{aligned} \tag{7}$$

where we have used the fact that $\| \boldsymbol{h}^*(\bar{\boldsymbol{e}}(t))\|$ satisfies analogous relation to (4), and $\forall \tau \geq 0\ s(\tau) \in (0,1]$. The right-hand side of (7) is finite for any $\tau < \infty$. Furthermore, for the asymptotic case with $\delta = 0$ taken in {17} one should consider now the integral at infinity

$$I(\tau_d,\infty) \leq (k_p+\eta)\int_{\tau_d}^{\infty} \| \bar{\boldsymbol{e}}^*(\xi)\|\, d\xi < \infty \tag{8}$$

which is finite by referring to similar reasoning as in (5)-(6). Hence, asymptotic convergence of joint angle (see Eq. {62})

$$\lim_{\tau\to\infty} \beta_N(\tau) = 2\arctan\left(B_{Nd}\cdot \exp\left(\frac{\sigma}{L_{hN}} I(\tau_d,\infty)\right)\right), \qquad (\sigma/L_{hN}) < 0 \tag{9}$$

to zero **is not possible** also in this case. However, according to (9) one may find that terminal value of β_N will be smaller for smaller value of offset $|L_{hN}|$. The above complementary analysis confirms and extends (for the case of $\delta = 0$) the statements included in Remark 2 in [1].

References

[1] M. Michałek. Application of the VFO method to set-point control for the N-trailer vehicle with off-axle hitching. *International Journal of Control*, 85(5):502–521, 2012.

[2] M. Michałek and K. Kozłowski. VFO tracking and set-point control for an articulated vehicle with off-axle hitched trailer. In *Proc. of the European Control Conference*, pages 266–271, Budapest, Hungary, 2009.

[3] M. Michałek and K. Kozłowski. Vector-Field-Orientation feedback control method for a differentially driven vehicle. *IEEE Transactions on Control Systems Technology*, 18(1):45–65, 2010.

□

[2]Henceforth, the notation and symbols are used according to [1].